The Canadian Mathematical Society
in collaboration with
The Center for Education in Mathematics and Computing

The First Canadian Open Mathematics Challenge

Wednesday, November 27, 1996

Examination Paper

TIME: $2\frac{1}{2}$ hours
Calculators are NOT permitted

Do not open this booklet until instructed to do so.
There are two parts to the paper.

PART A
This part of the paper consists of 10 questions, each worth 3 marks. You can earn full value for each question by entering the correct answer in the space provided. Any work you do in obtaining an answer will be considered for part marks if you do not have the correct answer, provided that it is done in the space allocated to that question in your answer booklet.

PART B
This part of the paper consists of 3 questions, each worth 10 marks. Finished solutions must be written in the appropriate location in the answer booklet. Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be provided by your supervising teacher. Any extra papers should be placed inside your answer booklet.
Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

Canadian Open Mathematics Challenge

NOTE: 1. Please read the instructions on the front cover of this booklet. 2. Write solutions in the answer booklet provided. 3. It is expected that all calculations and answers will be expressed as exact numbers such as $4\pi, 2+\sqrt{7}$, etc. 4. Calculators are **not** allowed.

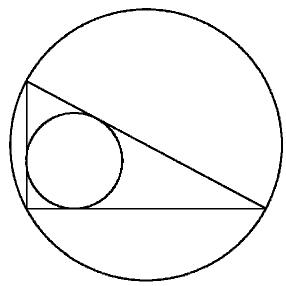
PART A

Answer all questions. The problems in this section are worth three marks each. A correct answer will receive full marks, but partial marks may be earned from any written work supplied.

1. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation $2x^3 + 9x^2 - 6x - 5 = 0$.
   What is the third root of the second equation?

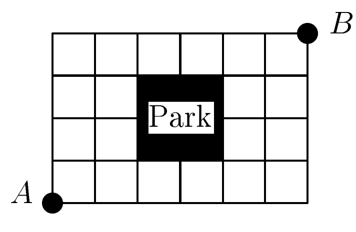
2. The numbers a, b, c are the digits of a three digit number which satisfy $49a+7b+c = 286$. What is the three digit number $(100a + 10b + c)$?

3. The vertices of a right–angled triangle are on a circle of radius R and the sides of the triangle are tangent to another circle of radius r. If the lengths of the sides about the right angle are 16 and 30, determine the value of $R + r$.



4. Determine the smallest positive integer, n, which satisfies the equation $n^3 + 2n^2 = b$, where b is the square of an odd integer.

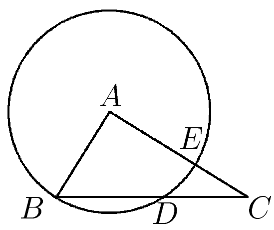
5. A road map of Grid City is shown in the diagram. The perimeter of the park is a road but there is no road through the park. How many different shortest road routes are there from point A to point B?



6. In a 14 team baseball league, each team played each of the other teams 10 times. At the end of the season, the number of games won by each team differed from those won by the

team that immediately followed it by the same amount. Determine the greatest number of games the last place team could have won, assuming that no ties were allowed.

7. Triangle ABC is right angled at A. The circle with center A and radius AB cuts BC and AC internally at D and E respectively. If $BD = 20$ and $DC = 16$, determine AC^2.



8. Determine all pairs of integers (x, y) which satisfy the equation

$$6x^2 - 3xy - 13x + 5y = -11.$$

9. If $\log_{2n}(1944) = \log_n\left(486\sqrt{2}\right)$, compute n^6.

10. Determine the sum of the angles A, B, where $0° \leq A, B \leq 180°$ and

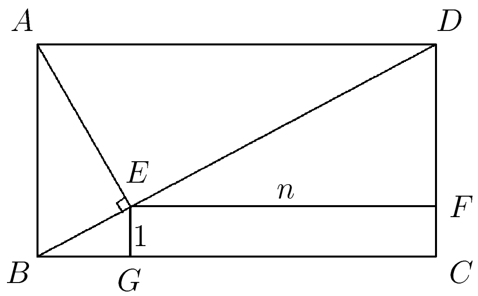
$$\sin A + \sin B = \sqrt{\frac{3}{2}}, \ \cos A + \cos B = \sqrt{\frac{1}{2}}.$$

PART B

Answer all questions. The problems in this section are worth 10 marks each. Marks will be based on presentation. A correct solution poorly presented will not earn full marks.

1. Three numbers form an arithmetic sequence, the common difference being 11. If the first number is decreased by 6, the second is decreased by 1 and the third is doubled, the resulting numbers are in geometric sequence. Determine the numbers which form the arithmetic sequence.

2. A rectangle $ABCD$ has diagonal of length d. The line AE is drawn perpendicular to the diagonal BD. The sides of the rectangle $EFCG$ have lengths n and 1. Prove $d^{2/3} = n^{2/3}+1$.



3. **(a)** Given positive numbers $a_1, a_2, a_3, \ldots, a_n$ and the quadratic function

$$f(x) = \sum_{i=1}^{n} (x - a_i)^2 \ ,$$

show that $f(x)$ attains its minimum value at $\dfrac{1}{n}\sum_{i=1}^{n} a_i$, and prove that

$$\sum_{i=1}^{n} {a_i}^2 \geq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \ .$$

(b) The sum of sixteen positive numbers is 100 and the sum of their squares is 1000. Prove that none of the sixteen numbers is greater than 25.