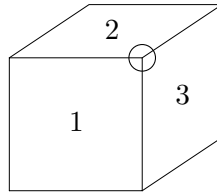


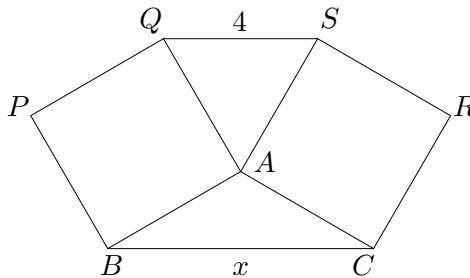
2011 COMC contest

Short Answer Problems

- A1.** If r is a number such that $r^2 - 6r + 5 = 0$, what is the value of $(r - 3)^2$?
- A2.** Carmen selects four different numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ whose sum is 11. If ℓ is the *largest* of these four numbers, what is the value of ℓ ?
- A3.** The faces of a cube contain the numbers 1, 2, 3, 4, 5, 6 such that the sum of the numbers on each pair of opposite faces is 7. For each of the cube's eight corners, we multiply the three numbers on the faces incident to that corner, and write down its value. (In the diagram, the value of the indicated corner is $1 \times 2 \times 3 = 6$.) What is the sum of the eight values assigned to the cube's corners?



- A4.** In the figure, $AQPB$ and $ASRC$ are squares, and AQS is an equilateral triangle. If $QS = 4$ and $BC = x$, what is the value of x ?



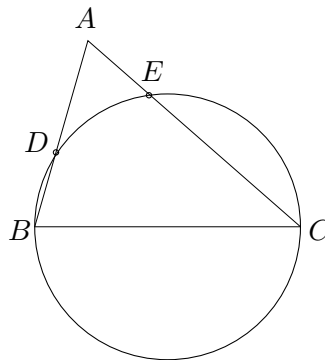
B1. Arthur is driving to David's house intending to arrive at a certain time. If he drives at 60 km/h, he will arrive 5 minutes late. If he drives at 90 km/h, he will arrive 5 minutes early. If he drives at n km/h, he will arrive exactly on time. What is the value of n ?

B2. Integers $a, b, c, d,$ and e satisfy the following three properties:

- (i) $2 \leq a < b < c < d < e < 100$
- (ii) $\gcd(a, e) = 1$
- (iii) a, b, c, d, e form a geometric sequence.

What is the value of c ?

B3. In the figure, BC is a diameter of the circle, where $BC = \sqrt{901}$, $BD = 1$, and $DA = 16$. If $EC = x$, what is the value of x ?

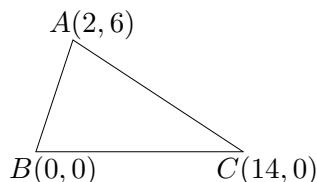


B4. A group of n friends wrote a math contest consisting of eight short-answer problems $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$, and four full-solution problems F_1, F_2, F_3, F_4 . Each person in the group correctly solved exactly 11 of the 12 problems. We create an 8×4 table. Inside the square located in the i^{th} row and j^{th} column, we write down the number of people who correctly solved both problem S_i and problem F_j . If the 32 entries in the table sum to 256, what is the value of n ?

	F_1	F_2	F_3	F_4
S_1				
S_2				
S_3				
S_4				
S_5				
S_6				
S_7				
S_8				

Full Solution Problems

C1. ABC is a triangle with coordinates $A = (2, 6)$, $B = (0, 0)$, and $C = (14, 0)$.



- (a) Let P be the midpoint of AB . Determine the equation of the line perpendicular to AB passing through P .
- (b) Let Q be the point on line BC for which PQ is perpendicular to AB . Determine the length of AQ .
- (c) There is a (unique) circle passing through the points A , B , and C . Determine the radius of this circle.
- C2.** Charlotte writes a test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. Charlotte's teacher announces that for every five *consecutive* questions on the test, the answers to *exactly* three of them are TRUE. Just before the test starts, the teacher whispers to Charlotte that the answers to the first and last questions are both FALSE.
- (a) Determine the number of questions for which the correct answer is TRUE.
- (b) What is the correct answer to the sixth question on the test?
- (c) Explain how Charlotte can correctly answer all 100 questions on the test.
- C3.** Let n be a positive integer. A row of $n + 1$ squares is written from left to right, numbered $0, 1, 2, \dots, n$, as shown.

0	1	2	\dots	n
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Two frogs, named Alphonse and Beryl, begin a race starting at square 0. For each second that passes, Alphonse and Beryl make a jump to the right according to the following rules: if there are at least eight squares to the right of Alphonse, then Alphonse jumps eight squares to the right. Otherwise, Alphonse jumps one square to the right. If there are at least seven squares to the right of Beryl, then Beryl jumps seven squares to the right. Otherwise, Beryl jumps one square to the right. Let $A(n)$ and $B(n)$ respectively denote the number of seconds for Alphonse and Beryl to reach square n . For example, $A(40) = 5$ and $B(40) = 10$.

- (a) Determine an integer $n > 200$ for which $B(n) < A(n)$.
- (b) Determine the largest integer n for which $B(n) \leq A(n)$.

C4. Let $f(x) = x^2 - ax + b$, where a and b are positive integers.

- (a) Suppose $a = 2$ and $b = 2$. Determine the set of real roots of $f(x) - x$, and the set of real roots of $f(f(x)) - x$.
- (b) Determine the number of pairs of positive integers (a, b) with $1 \leq a, b \leq 2011$ for which every root of $f(f(x)) - x$ is an integer.

COMC 2011 Solutions

A1. If r is a number for which $r^2 - 6r + 5 = 0$, what is the value of $(r - 3)^2$?

Solution: The answer is 4.

Solution 1: Note that $(r - 3)^2 = r^2 - 6r + 9$. Since $r^2 - 6r + 5 = 0$, $r^2 - 6r + 9 = 4$. Therefore, the answer is 4.

Solution 2: The quadratic equation $r^2 - 6r + 5$ factors as

$$(r - 1)(r - 5).$$

Therefore, $r = 1$ or $r = 5$. If $r = 1$, then $(r - 3)^2 = (-2)^2 = 4$. If $r = 5$, then $(r - 3)^2 = 2^2 = 4$. In either case, $(r - 3)^2 = 4$.

Solution 3: By completing the square on $r^2 - 6r + 5$, we have $r^2 - 6r + 5 = (r - 3)^2 - 4$. Since $r^2 - 6r + 5 = 0$, $(r - 3)^2 - 4 = 0$. Hence, $(r - 3)^2 = 4$.

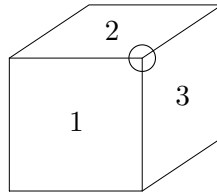
- A2.** Carmen selects four different numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ whose sum is 11. If ℓ is the *largest* of these four numbers, what is the value of ℓ ?

Solution: The answer is 5.

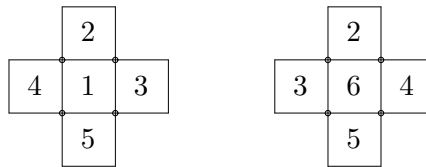
Solution 1: Note that the sum of the smallest four integers in the list is $1 + 2 + 3 + 4 = 10$. Hence, $1 + 2 + 3 + 5 = 11$. The largest positive integer in this sum is 5. Therefore, $\ell = 5$.

Solution 2: Since ℓ is the largest of four numbers from $\{1, 2, 3, 4, 5, 6, 7\}$, $\ell \geq 4$. Therefore, ℓ is equal to one of 4, 5, 6 and 7. If $\ell = 7$, then the smallest possible sum of the four numbers is $1 + 2 + 3 + 7 = 13 > 11$. Therefore, $\ell \neq 7$. Similarly, if $\ell = 6$, then the smallest possible sum of the four numbers is $1 + 2 + 3 + 6 = 12 > 11$. Similarly, $\ell \neq 4$. Therefore, $\ell = 5$.

A3. The faces of a cube contain the numbers 1, 2, 3, 4, 5, 6 such that the sum of the numbers on each pair of opposite faces is 7. For each of the cube's eight corners, we multiply the three numbers on the faces incident to that corner, and write down its value. (In the diagram, the value of the indicated corner is $1 \times 2 \times 3 = 6$.) What is the sum of the eight values assigned to the cube's corners?



Solution: The answer is 343.



Solution 1: The left picture shows the corners and the faces touching the side labeled 1 and the right picture shows the opposite side of the die, whose label is 6, which is incident to the other four corners.

We compute the eight numbers individually and sum the eight numbers. The eight triplets of integers at the eight corners are

$$(1, 2, 3), (1, 2, 4), (1, 3, 5), (1, 4, 5), (6, 2, 3), (6, 2, 4), (6, 3, 5), (6, 4, 5).$$

These eight values are

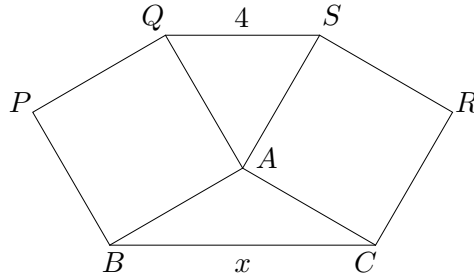
$$\begin{aligned}
 1 \times 2 \times 3 &= 6 \\
 1 \times 2 \times 4 &= 8 \\
 1 \times 3 \times 5 &= 15 \\
 1 \times 4 \times 5 &= 20 \\
 6 \times 2 \times 3 &= 36 \\
 6 \times 2 \times 4 &= 48 \\
 6 \times 3 \times 5 &= 90 \\
 6 \times 4 \times 5 &= 120 .
 \end{aligned}$$

The sum of these eight positive integers is 343.

Solution 2: Since no corner contains two numbers that sum to 7, the sum in solution 1 can be computed as

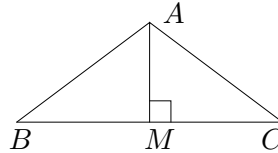
$$(1 + 6)(2 + 5)(3 + 4) = 7^3 = 343.$$

A4. In the figure, $AQPB$ and $ASRC$ are squares, and AQS is an equilateral triangle. If $QS = 4$ and $BC = x$, what is the value of x ?



Solution: The answer is $4\sqrt{3}$.

Solution 1: Since $\triangle AQS$ is equilateral, $AQ = QS = AS$. Since $QS = 4$, $AQ = AS = 4$. Since $AQPB$ and $ASRC$ are squares, $AB = AQ = 4$ and $AC = AS = 4$. Since $\triangle AQS$ is equilateral, $\angle QAS = 60^\circ$. Therefore, $\angle BAC = 360^\circ - 90^\circ - 90^\circ - \angle QAS = 120^\circ$.



Drop the perpendicular from A to side BC and let this perpendicular intersect BC at M . Then by symmetry, M is the midpoint of BC and $\angle BAM = \angle CAM = \angle BAC/2 = 120/2 = 60^\circ$. Therefore, $\triangle ABM$ is a $30 - 60 - 90$ triangle. Consequently,

$$\frac{BM}{BA} = \frac{\sqrt{3}}{2}.$$

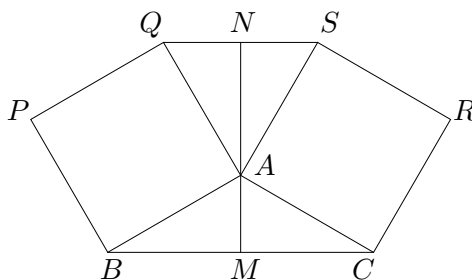
Hence, we have $BM = 4\sqrt{3}/2 = 2\sqrt{3}$. Similarly, $CM = 2\sqrt{3}$. Therefore, $BC = BM + CM = 4\sqrt{3}$.

Solution 2: By Solution 1, $AB = AC = 4$ and $\angle BAC = 120^\circ$. By the Cosine Law, we have

$$\begin{aligned} BC &= \sqrt{AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos \angle BAC} \\ &= \sqrt{4^2 + 4^2 - 2 \cdot 4 \cdot 4 \cdot \cos 120^\circ} \\ &= \sqrt{32 - 32 \cdot (-1/2)} = \sqrt{32 + 16} = \sqrt{48} = 4\sqrt{3}. \end{aligned}$$

Therefore, $x = 4\sqrt{3}$.

Solution 3:



Let M, N be the midpoints of BC and QS , respectively. By symmetry, M, A, N are collinear and the line MN is perpendicular to lines QS and BC . By Solution 1, $\angle QAS = 60^\circ$ and $\angle BAC = 120^\circ$. Therefore, by symmetry, $\angle QAN = 30^\circ$ and $\angle BAM = 60^\circ$. Since $\triangle AQS$ is equilateral, $\angle AQN = 60^\circ$ and $\angle ABM = 180^\circ - \angle BAM - \angle AMB = 180^\circ - 60^\circ - 90^\circ = 30^\circ$. Since $AB = AQ$, $\triangle ANQ$ is congruent to $\triangle BMA$. Therefore, $BM = AN$. By the Pythagorean Theorem,

$$BM = AN = \sqrt{AQ^2 - QN^2} = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}.$$

Hence, $x = BC = 2 \cdot BM = 2\sqrt{3}$.

- B1.** Arthur is driving to David's house intending to arrive at a certain time. If he drives at 60 km/h, he will arrive 5 minutes late. If he drives at 90 km/h, he will arrive 5 minutes early. If he drives at n km/h, he will arrive exactly on time. What is the value of n ?

The answer is 72.

Solution 1: Let d be the distance from Arthur to David's house in km and t the time, in hours, for Arthur to drive to David's place driving at n km/h. If he drives at 60 km/h, Arthur will drive for t hours + 5 minutes = $t + 5/60$ hours. If he drives at 90km/h, Arthur will drive for t hours - 5 minutes = $t - 5/60$ hours. Therefore, using the *distance = speed × time* formula, we have

$$d = nt = 60(t + 5/60) = 90(t - 5/60). \quad (1)$$

This simplifies to

$$d = nt = 60t + 5 = 90t - \frac{15}{2}, \quad (1)$$

We first determine t . Using the right-most equation of (1), we have $30t = 5 + \frac{15}{2} = \frac{25}{2}$. Therefore, $t = 25/60$. Hence, $d = 60t + 5 = 60(25/60) + 5 = 30$. Consequently, $n = d/t = 30/(25/60) = 30 \times 60/25 = 72$ km/h.

Solution 2: Let d be the distance from Arthur to David's house. Note that the time it takes for Arthur to drive to David's place at n km/h is the *average* of the times it take for Arthur to drive to David's place at 60 and 90 km/h, respectively. Hence,

$$\frac{d}{n} = \frac{\frac{d}{60} + \frac{d}{90}}{2}.$$

Dividing both sides by d and cross multiplying yields

$$\frac{2}{n} = \frac{1}{60} + \frac{1}{90} = \frac{5}{180}.$$

Hence, $5n = 360$. Therefore, $n = 72$.

B2. Integers $a, b, c, d,$ and e satisfy the following three properties:

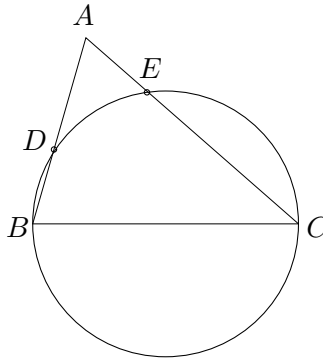
- (i) $2 \leq a < b < c < d < e < 100$
- (ii) $\gcd(a, e) = 1$
- (iii) a, b, c, d, e form a geometric sequence.

What is the value of c ?

Solution: The answer is 36.

Let r be the common ratio of the geometric sequence a, b, c, d, e . Since $a < b < c < d < e$, $r > 1$. Then $a = a, b = ar, c = ar^2, d = ar^3, e = ar^4$. Since a, e have no common factors and $a > 1$, r is not an integer. Let x/y be this common ratio, where x, y are positive integers and $\gcd(x, y) = 1$. Since $r > 1$ and is not an integer, $x > y > 1$. Therefore, $b = ax/y, c = ax^2/y^2, d = ax^3/y^3$ and $e = ax^4/y^4$. Since e is an integer and $\gcd(x, y) = 1$, a is divisible by y^4 . Then $a = ky^4$ for some positive integer k . Then $a = ky^4, b = kxy^3, c = kx^2y^2, d = kx^3y, e = kx^4$. Since $\gcd(a, e) = 1, k = 1$. Hence, $a = y^4$ and $e = x^4$. Since $2 \leq a < e < 100$ and $3^4 < 100 < 4^4, 2 \leq y < x \leq 3$, which implies that $x = 3$ and $y = 2$. Then $c = kx^2y^2 = 1 \cdot 3^2 \cdot 2^2 = 6^2 = 36$.

B3. In the figure, BC is a diameter of the circle, where $BC = \sqrt{901}$, $BD = 1$, and $DA = 16$. If $EC = x$, what is the value of x ?



Solution: The answer is 26.

Solution 1: Since BC is the diameter of the circle, $\angle BDC = \angle BEC = 90^\circ$. By the Pythagorean Theorem, we have

$$CD = \sqrt{BC^2 - BD^2} = \sqrt{901 - 1^2} = \sqrt{900} = 30.$$

Since $\angle BDC = 90^\circ$, $\angle ADC = 90^\circ$. Then by the Pythagorean Theorem, we have

$$AC = \sqrt{AD^2 + DC^2} = \sqrt{16^2 + 30^2} = 34.$$

Since $x = CE$, $AE = 34 - x$. We need to determine x . By the Pythagorean Theorem, we have $BE = \sqrt{BA^2 - AE^2} = \sqrt{BC^2 - CE^2}$. Hence,

$$BA^2 - AE^2 = BC^2 - CE^2.$$

Note that $BA = BD + DA = 16 + 1 = 17$. Therefore,

$$\begin{aligned} 17^2 - (34 - x)^2 &= 901 - x^2 \\ \Rightarrow x^2 + 289 &= (x - 34)^2 + 901 \\ \Rightarrow x^2 + 289 &= x^2 - 68x + 1156 + 901 \\ \Rightarrow 68x &= 1768. \end{aligned}$$

Therefore, $x = 1768/68 = (17 \times 104)/(17 \times 4) = 104/4 = 26$. Hence, $EC = 26$.

Solution 2: As in Solution 1, $\angle BDC = \angle BEC = 90^\circ$, $CD = 30$ and $AC = 34$. By computing the area of $\triangle ABC$ in two different ways, we have

$$\text{the area of } \triangle ABC = \frac{1}{2} \times AB \times DC = \frac{1}{2} \times AC \times BE.$$

Therefore, $AB \cdot DC = AC \cdot BE$. Hence, $17 \cdot 30 = 34 \cdot BE$. Therefore, $30 = 2 \cdot BE$. Equivalently, $BE = 15$. Therefore, by the Pythagorean Theorem,

$$EC = \sqrt{BC^2 - BE^2} = \sqrt{901 - 15^2} = \sqrt{676} = 26.$$

Solution 3: As in Solution 1, $\angle BDC = \angle BEC = 90^\circ$, $CD = 30$ and $AC = 34$. Compare $\triangle ADC$ and $\triangle AEB$. Note that $\angle ADC = \angle AEB = 90^\circ$ and $\angle DAC = \angle EAB$. Therefore, $\triangle ADC$ is similar to $\triangle AEB$. Consequently,

$$\frac{AD}{AE} = \frac{AC}{AB}.$$

Therefore, $AD \cdot AB = AE \cdot AC$. Note that $AB = AD + BD = 16 + 1 = 17$. Hence, $16 \times 17 = AE \cdot 34$. Therefore, $AE = 8$. We then conclude that $EC = AC - AE = 34 - 8 = 26$.

- B4.** A group of n friends wrote a math contest consisting of eight short-answer problems $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8$, and four full-solution problems F_1, F_2, F_3, F_4 . Each person in the group correctly solved exactly 11 of the 12 problems. We create an 8×4 table. Inside the square located in the i^{th} row and j^{th} column, we write down the number of people who correctly solved both problem S_i and problem F_j . If the 32 entries in the table sum to 256, what is the value of n ?

	F_1	F_2	F_3	F_4
S_1				
S_2				
S_3				
S_4				
S_5				
S_6				
S_7				
S_8				

Solution: The answer is 10.

Solution 1: The sum of all of the numbers written is the sum of all of the number of pairs of problems of the form (S_i, F_j) each student solved. The contribution of each student to this sum is the product of the number of problems this student solved in the short-answer part and the number of problems this student solved in the full-solution part. Since each student solved 11 problems, each student solved either 8 short-answer problems and 3 full-solution problems, or 7 short-answer problems and 4 full-solution problems. Let x be the number of students who solved 8 short-answer problems and 3 full-solution problems and y the number of students who solved 7 short-answer problems and 4 full-solution problems. Then the sum of the numbers written down is $8 \times 3 \times x + 7 \times 4 \times y = 256$. Hence, $24x + 28y = 256$. Dividing both sides by 4 yields $6x + 7y = 64$. Note that $0 \leq x \leq 10$. Substituting each such value of x , we get the following values of y :

x	y
0	$64/7$
1	$58/7$
2	$52/7$
3	$46/7$
4	$40/7$
5	$34/7$
6	4
7	$22/7$
8	$16/7$
9	$10/7$
10	$4/7$

We note that only $(x, y) = (6, 4)$ yields a non-negative integer solution for x and y . Hence, the number of students is $x + y = 6 + 4 = 10$.

Solution 2: Since each person solved 11 of the 12 problems, there was one problem that each person did not correctly solve. Let s_i be the number of people who missed problem S_i (for $i = 1, \dots, 8$) and let f_j be the number of people who missed problem F_j (for $j = 1, \dots, 4$).

As in Solution 1, let x be the number of students who solved 8 short-answer problems and 3 full-solution problems, and let y be the number of students who solved 7 short-answer problems and 4 full-solution problems. By definition, $y = s_1 + s_2 + \dots + s_8$ and $x = f_1 + f_2 + f_3 + f_4$ and $n = x + y$.

Consider the entry in the i^{th} row and j^{th} column of our 8×4 table. This number must be $n - s_i - f_j$. Adding all 32 entries, we find that $256 = 32n - 4(s_1 + \dots + s_8) - 8(f_1 + \dots + f_4) = 32n - 4y - 8x = 32(x + y) - 4y - 8x = 24x + 28y$. Therefore, $24x + 28y = 256$. We then complete the problem as in Solution 1.

Solution 3: Let s_i, f_j be as in Solution 2. Then

$$n = (s_1 + s_2 + \dots + s_8) + (f_1 + f_2 + f_3 + f_4).$$

Also as in Solution 2, we have

$$\begin{aligned} 256 &= 32n - 4(s_1 + s_2 + \dots + s_8) - 8(f_1 + f_2 + f_3 + f_4) \\ &= 32n - 4n - 4(f_1 + f_2 + f_3 + f_4) \\ &= 28n - 4(f_1 + f_2 + f_3 + f_4) \end{aligned}$$

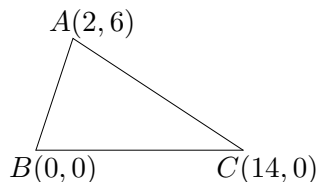
Therefore, $64 = 7n - (f_1 + f_2 + f_3 + f_4)$. Hence, $n \geq 10$. But note that if $n \geq 11$, then

$$(f_1 + f_2 + f_3 + f_4) = 7n - 64 = n + (6n - 64) > n.$$

Since $f_1 + f_2 + f_3 + f_4$ is the number of people that missed a full-solution problem, $f_1 + f_2 + f_3 + f_4$ is at most the number of people in the group, which is n . This contradicts $f_1 + f_2 + f_3 + f_4 > n$. Hence, $n \not\geq 11$. This result in conjunction with $n \geq 10$ yields $n = 10$.

1 Full Solution Problems

C1. ABC is a triangle with coordinates $A = (2, 6)$, $B = (0, 0)$, and $C = (14, 0)$.



- Let P be the midpoint of AB . Determine the equation of the line perpendicular to AB passing through P .
- Let Q be the point on line BC for which PQ is perpendicular to AB . Determine the length of AQ .
- There is a (unique) circle passing through the points A , B , and C . Determine the radius of this circle.

Solution:

(a) The answer is $y = -1/3 \cdot x + 10/3$ or $x + 3y = 10$.

The midpoint of AB is

$$P = \left(\frac{0+2}{2}, \frac{0+6}{2} \right) = (1, 3).$$

The slope of AB is $6/2 = 3$. Therefore, the slope of the line perpendicular to AB is $-1/3$. Hence, the equation of the line perpendicular to AB passing through P is

$$y - 3 = \frac{-1}{3}(x - 1).$$

This is equivalent to

$$y = \frac{-1}{3}x + \frac{10}{3}.$$

Rewriting this yields

$$x + 3y = 10.$$

(b) The answer is 10.

Solution 1: The line BC is the line $y = 0$. Since Q lies on BC , the y -coordinate of Q is 0. Since Q also lies on the line passing through P perpendicular to AB and the equation of this line is $x + 3y = 10$, we substitute $y = 0$ into $x + 3y = 10$ to yield $x = 10$. Hence, $Q = (10, 0)$. Since $A = (2, 6)$, by the Pythagorean Theorem,

$$AQ = \sqrt{(10-2)^2 + (0-6)^2} = \sqrt{8^2 + 6^2} = 10.$$

Solution 2: As in Solution 1, $Q = (10, 0)$. Since Q lies on the perpendicular bisector of AB , $QA = QB$. Since $Q = (10, 0)$ and $B = (0, 0)$, $QA = QB = 10$.

(c) The answer is $5\sqrt{2}$ or $\sqrt{50}$.

Solution 1: Let $O = (x, y)$ be the centre of the circle. Since O lies on the perpendicular bisector of BC , $x = (0 + 14)/2 = 7$. Since O lies on the line perpendicular to AB passing through P and the equation of the line passing through P perpendicular to AB is $x + 3y = 10$, we substitute $x = 7$ into $x + 3y = 10$ to yield $y = 1$. Hence, the centre of the circle is at $(7, 1)$. The radius of the circle is the distance from O to any of A, B, C . For simplicity's sake, we compute the length of OB , since $B = (0, 0)$. By the Pythagorean Theorem, the radius of the circle is $OB = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$.

Solution 2: We will use the following property of a triangle; let a, b, c be the side lengths of a triangle, R the circumradius of the triangle and K the area of the triangle. Then the quantities a, b, c, R, K have the following relationship;

$$K = \frac{abc}{4R}.$$

In this triangle, $AB = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$, $BC = 14$ and $CA = \sqrt{(14 - 2)^2 + 6^2} = \sqrt{180} = 3\sqrt{20}$. Note that

$$K = \frac{1}{2} \times BC \times \{\text{the height to side BC}\} = \frac{1}{2} \times 14 \times 6 = 42.$$

Therefore,

$$R = \frac{AB \cdot BC \cdot CA}{4K} = \frac{2\sqrt{10} \times 14 \times 3\sqrt{20}}{4 \times 42} = \frac{60\sqrt{2} \times 14}{4 \times 42} = 5\sqrt{2}.$$

- C2.** Charlotte writes a test consisting of 100 questions, where the answer to each question is either TRUE or FALSE. Charlotte's teacher announces that for every five *consecutive* questions on the test, the answers to *exactly* three of them are TRUE. Just before the test starts, the teacher whispers to Charlotte that the answers to the first and last questions are both FALSE.
- Determine the number of questions for which the correct answer is TRUE.
 - What is the correct answer to the sixth question on the test?
 - Explain how Charlotte can correctly answer all 100 questions on the test.

Solution

- The answer is 60.

Split the 100 problems into groups of 5, namely $1-5, 6-10, 11-15, \dots, 91-95, 96-100$. Since there are 100 problems and five problems per group and every set of five consecutive problems contain exactly three problems whose answer is TRUE, each group contains three problems whose answers are TRUE. Since there are 20 groups, there are $20 \times 3 = 60$ problems whose answers are TRUE on the test.

- Consider the problems 1, 2, 3, 4, 5, 6. Among problems 1–5, there are exactly three problems whose answer is TRUE. Since the answer to the first problem is FALSE, among problems 2–5, exactly three of these problems have answer TRUE. Now consider problem 6. Since problems 2–6 contains exactly three problems whose answers are TRUE and problems 2–5 already contain 3 such problems, the answer to problem 6 is FALSE.
- Solution 1:** We claim that the answer to problem n has the same answer as problem $n + 5$. Consider the problems $n, n + 1, n + 2, n + 3, n + 4, n + 5$. Note that problems $n, n + 1, n + 2, n + 3, n + 4$ contain three problems whose answers are TRUE and problems $n + 1, n + 2, n + 3, n + 4, n + 5$ contain three problems whose answers are TRUE. Note that problems $n + 1, n + 2, n + 3, n + 4$ contain either 2 or 3 problems whose answers are TRUE. In the former case, the answers to both problem n and problem $n + 5$ are TRUE. In the latter case, the answers to both problem n and problem $n + 5$ are FALSE. In either case, problems n and $n + 5$ have the same answer.

By this claim, problems $\{1, 6, 11, 16, \dots, 91, 96\}$ have the same answers. So do $\{2, 7, 12, 17, \dots, 92, 97\}$, $\{3, 8, 13, 18, \dots, 93, 98\}$, $\{4, 9, 14, 19, \dots, 94, 99\}$ and $\{5, 10, 15, 20, \dots, 95, 100\}$. For each of these five groups of problems, if we can determine the answer to one problem in the group, we can determine the answers to every problem in the group. Since the answer to problem 1 is FALSE, the answers to problems $\{1, 6, 11, 16, \dots, 91, 96\}$ are all FALSE. Since problem 100 is FALSE, then the answers to problems $\{5, 10, 15, 20, \dots, 95, 100\}$ are also FALSE. Since problems 1 and 5 have answers FALSE, and

exactly three of problems 1, 2, 3, 4, 5 have answer TRUE, problems 2, 3, 4 have answer TRUE. Therefore, the answers to the remaining problems $\{2, 7, 12, 17, \dots, 92, 97\}$, $\{3, 8, 13, 18, \dots, 93, 98\}$, $\{4, 9, 14, 19, \dots, 94, 99\}$ are all TRUE. Having determined the correct answer to each question, Charlotte achieves a perfect score by answering FALSE, TRUE, TRUE, TRUE, FALSE to the first five questions, and repeating this pattern for each block of five consecutive questions.

Solution 2: As in Solution 1, problems $\{1, 6, 11, \dots, 96\}$ and $\{5, 10, 15, \dots, 100\}$ have answers FALSE. There are 40 such problems. By part (a), 60 of the 100 problems have answer TRUE. Therefore, the remaining 60 problems, mainly, $\{2, 7, \dots, 97\}$, $\{3, 8, \dots, 98\}$ and $\{4, 9, \dots, 99\}$, have answer FALSE.

Comment: An analogue solution to this solution is to define a variable x_i for Problem i , with $x_i = 1$ if the answer to Problem i is TRUE and $x_i = 0$ if the answer to Problem i is FALSE. Then based on the information given we have the following system of equations:

$$x_j + x_{j+1} + x_{j+2} + x_{j+3} + x_{j+4} = 3, \quad \forall 1 \leq j \leq 96$$

and $x_1 = 0$ and $x_{100} = 0$. Charlotte needs to determine all x_i where $1 \leq i \leq 100$. Since $x_j \in \{0, 1\}$, solving this system of the equations yield

$$\begin{aligned} x_1 = x_6 = x_{11} = x_{16} = \dots = x_{91} = x_{96} &= 0 \\ x_2 = x_7 = x_{12} = x_{17} = \dots = x_{92} = x_{97} &= 1 \\ x_3 = x_8 = x_{13} = x_{18} = \dots = x_{93} = x_{98} &= 1 \\ x_4 = x_9 = x_{14} = x_{19} = \dots = x_{94} = x_{99} &= 1 \\ x_5 = x_{10} = x_{15} = x_{20} = \dots = x_{95} = x_{100} &= 0 \end{aligned}$$

- C3.** Let n be a positive integer. A row of $n + 1$ squares is written from left to right, numbered $0, 1, 2, \dots, n$, as shown.

0	1	2	...	n
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Two frogs, named Alphonse and Beryl, begin a race starting at square 0. For each second that passes, Alphonse and Beryl make a jump to the right according to the following rules: if there are at least eight squares to the right of Alphonse, then Alphonse jumps eight squares to the right. Otherwise, Alphonse jumps one square to the right. If there are at least seven squares to the right of Beryl, then Beryl jumps seven squares to the right. Otherwise, Beryl jumps one square to the right. Let $A(n)$ and $B(n)$ respectively denote the number of seconds for Alphonse and Beryl to reach square n . For example, $A(40) = 5$ and $B(40) = 10$.

- (a) Determine an integer $n > 200$ for which $B(n) < A(n)$.
 (b) Determine the largest integer n for which $B(n) \leq A(n)$.

Solution 1: Note that if we write $n = 8q_1 + r_1$ where q_1, r_1 are non-negative integers and $0 \leq r_1 < 8$, then Alphonse performs q_1 8-square jumps and r_1 1-square jump. Then the number of jumps Alphonse performs is $A(n) = q_1 + r_1$. Similarly, if we write $n = 7q_2 + r_2$ where q_2, r_2 are non-negative integers and $0 \leq r_2 < 7$, then $B(n) = q_2 + r_2$.

- (a) Since Alphonse's 8-square jump is farther than Beryl's 7-square jump, in order for Beryl to finish faster than Alphonse, n must be an integer such that Beryl performs very few 1-square jumps and Alphonse performs many 1-square jumps, i.e. n should be an integer that is divisible by 7 and has a high remainder upon division by 8, i.e. 7. Note that 7 is such an integer. Note that adding $7 \times 8 = 56$ repeatedly to 7 preserves this property, i.e. 63, 119, 175, 231. Since $231 = 33 \times 7$, $B(231) = 33$. Since $231 = 28 \times 8 + 7$, Alphonse performs $28 + 7 = 35$ jumps, i.e. $A(231) = 35$. Therefore, $B(231) < A(231)$. Hence, $n = 231$ is such a positive integer.
- (b) Since $B(n) \leq A(n)$, we have $q_2 + r_2 \leq q_1 + r_1$. Since $8q_1 + r_1 = 7q_2 + r_2$ and $r_2 \leq q_1 + r_1 - q_2$,

$$8q_1 + r_1 \leq 7q_2 + q_1 + r_1 - q_2.$$

Equivalently, $7q_1 \leq 6q_2$. Therefore, $q_2 \geq 7q_1/6$. Substituting this into $8q_1 + r_1 = 7q_2 + r_2$ yields

$$8q_1 + r_1 \geq \frac{49}{6}q_1 + r_2.$$

Therefore,

$$\frac{q_1}{6} \leq r_1 - r_2.$$

Since $r_1 \leq 7$ and $r_2 \geq 0$, $r_1 - r_2 \leq 7$, which implies that $q_1 \leq 42$. Since $r_1 \leq 7$, $n = 8q_1 + r_1 \leq 8 \times 42 + 7 = 343$.

To prove that 343 is indeed the maximum, note that $343 = 42 \times 8 + 7$, which implies that $A(343) = 42 + 7 = 49$. Also, note that $343 = 49 \times 7$, which implies that $B(343) = 49$. Therefore, $A(343) = B(343)$. Hence, $n = 343$ is the maximum positive integer such that $B(n) \leq A(n)$.

Solution 2:

Using the notation in Solution 1, we have $A(n) = q_1 + r_1$ and $B(n) = q_2 + r_2$. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x . For example, $\lfloor \frac{23}{8} \rfloor = 2$. Note that $q = \lfloor n/8 \rfloor$. Then $r = n - 8q = n - 8\lfloor n/8 \rfloor$. Hence, $A(n)q + r = n - 7\lfloor n/8 \rfloor$. Similarly, $B(n) = n - 6\lfloor n/7 \rfloor$.

- (a) We seek an integer $n > 200$ for which $B(n) = n - 6\lfloor \frac{n}{7} \rfloor < n - 7\lfloor \frac{n}{8} \rfloor = A(n)$, i.e., $7\lfloor \frac{n}{8} \rfloor < 6\lfloor \frac{n}{7} \rfloor$. If we were to remove the floor notation, the inequality would reduce to $\frac{7n}{8} < \frac{6n}{7}$, which is not true. Thus, in order to achieve the inequality $7\lfloor \frac{n}{8} \rfloor < 6\lfloor \frac{n}{7} \rfloor$, we want to make $\frac{n}{8} - \lfloor \frac{n}{8} \rfloor$ as large as possible and $\frac{n}{7} - \lfloor \frac{n}{7} \rfloor$ as small as possible. One way to achieve this is to make $\frac{n}{8}$ just less than an integer, so that $\lfloor \frac{n}{8} \rfloor$ will be approximately $\frac{n}{8} - 1$, while ensuring that $\frac{n}{7}$ is equal to an integer, so that $\lfloor \frac{n}{7} \rfloor = \frac{n}{7}$.

Let $n = 56k + 7$, for some integer $k > 0$. Then $\lfloor \frac{n}{8} \rfloor = \lfloor \frac{56k+7}{8} \rfloor = \lfloor 7k + \frac{7}{8} \rfloor = 7k$, and $\lfloor \frac{n}{7} \rfloor = \lfloor \frac{56k+7}{7} \rfloor = 8k + 1$. Then our inequality becomes $7 \cdot 7k < 6 \cdot (8k + 1)$, which is equivalent to $k < 6$. For example, if $k = 4$, then $n = 56 \cdot 4 + 7 = 231$ is an integer satisfying $7\lfloor \frac{n}{8} \rfloor < 6\lfloor \frac{n}{7} \rfloor$, which implies that $B(231) < A(231)$. Checking, we see that $A(231) = 231 - 7\lfloor \frac{231}{8} \rfloor = 35$ and $B(231) = 231 - 6\lfloor \frac{231}{7} \rfloor = 33$. Thus, $n = 231$ is indeed a solution to the problem. Another solution is $n = 56 \cdot 5 + 7 = 287$, found by letting $k = 5$. Other solutions include $n = 238$ and $n = 239$.

- (b) For each positive integer n , there exist unique integers p, q, r for which $n = 56p + 8q + r$, where $0 \leq q \leq 6$ and $0 \leq r \leq 7$. The inequality $B(n) \leq A(n)$ is equivalent to $7\lfloor \frac{n}{8} \rfloor \leq 6\lfloor \frac{n}{7} \rfloor$.

We have $\lfloor \frac{n}{8} \rfloor = \lfloor \frac{56p+8q+r}{8} \rfloor = 7p + q + \lfloor \frac{r}{8} \rfloor = 7p + q$, since $0 \leq r \leq 7$. And also we have $\lfloor \frac{n}{7} \rfloor = \lfloor \frac{56p+8q+r}{7} \rfloor = 8p + q + \lfloor \frac{q+r}{7} \rfloor$.

Thus, the inequality $7\lfloor \frac{n}{8} \rfloor \leq 6\lfloor \frac{n}{7} \rfloor$ is equivalent to $7(7p + q) \leq 6(8p + q) + 6\lfloor \frac{q+r}{7} \rfloor$, which simplifies to $p + q \leq 6\lfloor \frac{q+r}{7} \rfloor$. Since $q + r \leq 6 + 7 = 13$, we must have $p + q \leq 6\lfloor \frac{13}{7} \rfloor = 6$.

We wish to determine the largest integer $n = 56p + 8q + r$ for which the above inequality is satisfied. To do this, we want to maximize p . Since $p + q \leq 6$, let us first try $p = 6$. Then this forces $q = 0$. This case satisfies the inequality $p + q \leq 6\lfloor \frac{q+r}{7} \rfloor$ if and only if

$r = 7$. We remark that the triplet $(p, q, r) = (6, 0, 7)$ yields $n = 56 \times 6 + 7 = 343$ which is indeed a solution because $A(343) = 343 - 7 \lfloor \frac{343}{8} \rfloor = 49$ and $B(343) = 343 - 6 \lfloor \frac{343}{7} \rfloor = 49$.

To show that $n = 343$ is indeed the largest value of n satisfying $B(n) \leq A(n)$, we note that $(p, q, r) = (6, 0, 7)$ is the only triplet satisfying the inequality for $p = 6$, from the analysis above. And so any other solution must have $p \leq 5$. But then such a solution would have $n = 56p + 8q + r \leq 56 \times 5 + 8 \times 6 + 7 = 335 < 343$.

C4. Let $f(x) = x^2 - ax + b$, where a and b are positive integers.

- (a) Suppose $a = 2$ and $b = 2$. Determine the set of real roots of $f(x) - x$, and the set of real roots of $f(f(x)) - x$.
- (b) Determine the number of pairs of positive integers (a, b) with $1 \leq a, b \leq 2011$ for which every root of $f(f(x)) - x$ is an integer.

Solution:

(a) If $a = 2$ and $b = 2$, then $f(x) = x^2 - 2x + 2$. Hence, $f(x) - x = x^2 - 3x + 2 = (x - 2)(x - 1)$. Therefore, the roots of $f(x) - x$ are 1 and 2.

We now determine $f(f(x)) - x$. Note that $f(f(x)) = (x^2 - 2x + 2)^2 - 2(x^2 - 2x + 2) + 2 = x^4 - 4x^3 + 6x^2 - 4x + 2$. Therefore,

$$f(f(x)) - x = x^4 - 4x^3 + 6x^2 - 5x + 2.$$

Note that 1 is a root of $f(f(x)) - x$. Then

$$x^4 - 4x^3 + 6x^2 - 5x + 2 = (x - 1)(x^3 - 3x^2 + 3x - 2) = (x - 1)(x - 2)(x^2 - x + 1).$$

Note that $x^2 - x + 1$ has no real roots since its discriminant is $1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$. Therefore, the real roots of $f(f(x)) - x$ are 1 and 2.

(b) The answer is 43.

First, we claim that if r is a root of $f(x) - x$, then r is a root of $f(f(x)) - x$. Since r is a root of $f(x) - x$, $f(r) - r = 0$, i.e. $f(r) = r$. Therefore,

$$f(f(r)) - r = f(r) - r = 0.$$

Hence, any root of $f(x) - x$ is a root of $f(f(x)) - x$. Consequently, $f(x) - x$ is a factor of $f(f(x)) - x$.

Note that $f(f(x)) - x = f(x^2 - ax + b) - x = (x^2 - ax + b)^2 - a(x^2 - ax + b) + b - x$,

$$= x^4 - 2ax^3 + (a^2 + 2b - a)x^2 - (2ab - a^2 + 1)x + (b^2 - ab + b).$$

Since $f(x) - x = x^2 - (a + 1)x + b$, $f(f(x)) - x$ factors as

$$f(f(x)) - x = (x^2 - (a + 1)x + b)(x^2 - (a - 1)x + (b - a + 1)).$$

Since both factors are monic, every root of $f(f(x)) - x$ is an integer if and only if the discriminants of both of these quadratic factors are perfect squares. These two discriminants are

$$(a + 1)^2 - 4b = a^2 + 2a + 1 - 4b$$

and

$$(a-1)^2 - 4(b-a+1) = a^2 + 2a + 1 - 4b - 4.$$

The first discriminant is four larger than the second discriminant. The only two perfect squares that differ by 4 are 4 and 0. This statement is true since if r, s are non-negative integers such that $r^2 - s^2 = 4$, then $(r-s)(r+s) = 4$. Since r, s are non-negative, $(r-s, r+s) = (2, 2)$ or $(1, 4)$. In the latter case, $r-s = 1$ and $r+s = 4$. Therefore, $r = 5/2$ and $s = 3/2$, which are not integers. Therefore, $(r-s, r+s) = (2, 2)$, i.e. $r = 2, s = 0$. Hence, the larger perfect square is $2^2 = 4$ and the smaller perfect square is 0.

Therefore, $a^2 + 2a + 1 - 4b = 4$. Rearranging this and factoring yields

$$(a+1)^2 = 4(b+1).$$

Since $(a+1)^2$ and 4 are perfect squares, $b+1$ is a perfect square. Therefore, there exists a positive integer m such that $b+1 = m^2$. Then $b = m^2 - 1$. Consequently, $(a+1)^2 = 4m^2$. Since a is a positive integer, $a+1 = 2m$. Hence, $a = 2m - 1$. Therefore, $(a, b) = (2m - 1, m^2 - 1)$.

We now verify that all such (a, b) have the property that the roots of $x^2 - (a+1)x + b$ and $x^2 - (a-1)x + (b-a+1)$ are all integers, implying that every root of $f(f(x)) - x$ is an integer. Substituting $(a, b) = (2m - 1, m^2 - 1)$ into these two polynomials yield $x^2 - 2mx + m^2 - 1 = (x - (m-1))(x - (m+1))$ and $x^2 - (2m-2)x + (m^2 - 2m + 1) = (x - (m-1))(x - (m-1))$. Since m is a positive integer, all four roots of $f(f(x)) - x$ is an integer. (Alternately, note that since the leading coefficient of each of the quadratic factors is 1, the roots of the quadratic factors are all integers if and only if the discriminant of the quadratic factors are both perfect square.)

Since $1 \leq a, b \leq 2011$, it remains to find the number of positive integers m such that $1 \leq 2m - 1, m^2 - 1 \leq 2011$. Since $1 \leq m^2 - 1 \leq 2011$, $2 \leq m^2 \leq 2012$. Hence, $2 \leq m \leq \lfloor \sqrt{2012} \rfloor = 44$, where $\lfloor t \rfloor$ denotes the largest integer less than or equal to t . There are 43 solutions for m , namely $m = 2, 3, \dots, 44$. These values of m clearly satisfy $1 \leq 2m - 1 \leq 2011$.

Therefore, the number of ordered positive integer pairs (a, b) that results in $f(f(x)) - x$ having all integer roots is 43.