2017 Canadian Mathematical Olympiad

Official Solutions

1. Let a, b, and c be non-negative real numbers, no two of which are equal. Prove that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2}>2.$$

Solution: The left-hand side is symmetric with respect to a, b, c. Hence, we may assume that $a > b > c \geq 0$. Note that replacing (a, b, c) with $(a-c, b-c, 0)$ lowers the value of the left-hand side, since the numerators of each of the fractions would decrease and the denominators remain the same. Therefore, to obtain the minimum possible value of the left-hand side, we may assume that $c = 0$.

Then the left-hand side becomes

$$\frac{a^2}{b^2}+\frac{b^2}{a^2},$$

which yields, by the Arithmetic Mean - Geometric Mean Inequality,

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \;\geq\; 2\sqrt{\frac{a^2}{b^2}\cdot\frac{b^2}{a^2}} \;=\; 2,$$

with equality if and only if $a^2/b^2 = b^2/a^2$, or equivalently, $a^4 = b^4$. Since $a, b \geq 0$, $a = b$. But since no two of a, b, c are equal, $a \neq b$. Hence, equality cannot hold. This yields

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} \;>\; 2.$$

Ultimately, this implies the desired inequality. □

Alternate solution: First, show that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2} - 2 \;=\; \frac{[a(a-b)(a-c)+b(b-a)(b-c)+c(c-a)(c-b)]^2}{[(a-b)(b-c)(c-a)]^2}.$$

Then Schur's Inequality tells us that the numerator of the right-hand side cannot be zero. □

2. Let f be a function from the set of positive integers to itself such that, for every n, the number of positive integer divisors of n is equal to $f(f(n))$. For example, $f(f(6)) = 4$ and $f(f(25)) = 3$. Prove that if p is prime then $f(p)$ is also prime.

   Solution: Let $d(n) = f(f(n))$ denote the number of divisors of n and observe that $f(d(n)) = f(f(f(n))) = d(f(n))$ for all n. Also note that because all divisors of n are distinct positive integers between 1 and n, including 1 and n, and excluding $n-1$ if $n > 2$, it follows that $2 \leq d(n) < n$ for all $n > 2$. Furthermore $d(1) = 1$ and $d(2) = 2$.

   We first will show that $f(2) = 2$. Let $m = f(2)$ and note that $2 = d(2) = f(f(2)) = f(m)$. If $m \geq 2$, then let m_0 be the smallest positive integer satisfying that $m_0 \geq 2$ and $f(m_0) = 2$. It follows that $f(d(m_0)) = d(f(m_0)) = d(2) = 2$. By the minimality of m_0, it follows that $d(m_0) \geq m_0$, which implies that $m_0 = 2$. Therefore if $m \geq 2$, it follows that $f(2) = 2$. It suffices to examine the case in which $f(2) = m = 1$. If $m = 1$, then $f(1) = f(f(2)) = 2$ and furthermore, each prime p satisfies that $d(f(p)) = f(d(p)) = f(2) = 1$ which implies that $f(p) = 1$. Therefore $d(f(p^2)) = f(d(p^2)) = f(3) = 1$ which implies that $f(p^2) = 1$ for any prime p. This implies that $3 = d(p^2) = f(f(p^2)) = f(1) = 2$, which is a contradiction. Therefore $m \neq 1$ and $f(2) = 2$.

   It now follows that if p is prime then $2 = f(2) = f(d(p)) = d(f(p))$ which implies that $f(p)$ is prime. □

   Remark. Such a function exists and can be constructed inductively.

3. Let n be a positive integer, and define $S_n = \{1, 2, \ldots, n\}$. Consider a non-empty subset T of S_n. We say that T is balanced if the median of T is equal to the average of T. For example, for $n = 9$, each of the subsets $\{7\}$, $\{2, 5\}$, $\{2, 3, 4\}$, $\{5, 6, 8, 9\}$, and $\{1, 4, 5, 7, 8\}$ is balanced; however, the subsets $\{2, 4, 5\}$ and $\{1, 2, 3, 5\}$ are not balanced. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.
(To define the median of a set of k numbers, first put the numbers in increasing order; then the median is the middle number if k is odd, and the average of the two middle numbers if k is even. For example, the median of $\{1, 3, 4, 8, 9\}$ is 4, and the median of $\{1, 3, 4, 7, 8, 9\}$ is $(4 + 7)/2 = 5.5$.)

Solution: The problem is to prove that there is an odd number of nonempty subsets T of S_n such that the average $A(T)$ and median $M(T)$ satisfy $A(T) = M(T)$. Given a subset T, consider the subset $T^* = \{n + 1 - t : t \in T\}$. It holds that $A(T^*) = n + 1 - A(T)$ and $M(T^*) = n + 1 - M(T)$, which implies that if $A(T) = M(T)$ then $A(T^*) = M(T^*)$. Pairing each set T with T^* yields that there are an even number of sets T such that $A(T) = M(T)$ and $T \neq T^*$.

Thus it suffices to show that the number of nonempty subsets T such that $A(T) = M(T)$ and $T = T^*$ is odd. Now note that if $T = T^*$, then $A(T) = M(T) = \frac{n+1}{2}$. Hence it suffices to show the number of nonempty subsets T with $T = T^*$ is odd. Given such a set T, let T' be the largest nonempty subset of $\{1, 2, \ldots, \lceil n/2 \rceil\}$ contained in T. Pairing T with T' forms a bijection between these sets T and the nonempty subsets of $\{1, 2, \ldots, \lceil n/2 \rceil\}$. Thus there are $2^{\lceil n/2 \rceil} - 1$ such subsets, which is odd as desired. □

Alternate solution: Using the notation from the above solution: Let B be the number of subsets T with $M(T) > A(T)$, C be the number with $M(T) = A(T)$, and D be the number with $M(T) < A(T)$. Pairing each set T counted by B with $T^* = \{n + 1 - t : t \in T\}$ shows that $B = D$. Now since $B + C + D = 2^n - 1$, we have that $C = 2^n - 1 - 2B$, which is odd.

4. Points P and Q lie inside parallelogram $ABCD$ and are such that triangles ABP and BCQ are equilateral. Prove that the line through P perpendicular to DP and the line through Q perpendicular to DQ meet on the altitude from B in triangle ABC.

   Solution: Let $\angle ABC = m$ and let O be the circumcenter of triangle DPQ. Since P and Q are in the interior of $ABCD$, it follows that $m = \angle ABC > 60^\circ$ and $\angle DAB = 180^\circ - m > 60^\circ$ which together imply that $60^\circ < m < 120^\circ$. Now note that $\angle DAP = \angle DAB - 60^\circ = 120^\circ - m$, $\angle DCQ = \angle DCB - 60^\circ = 120^\circ - m$ and that $\angle PBQ = 60^\circ - \angle ABQ = 60^\circ - (\angle ABC - 60^\circ) = 120^\circ - m$. This combined with the facts that $AD = BQ = CQ$ and $AP = BP = CD$ implies that triangles DAP, QBP and QCD are congruent. Therefore $DP = PQ = DQ$ and triangle DPQ is equilateral. This implies that $\angle ODA = \angle PDA + 30^\circ = \angle DQC + 30^\circ = \angle OQC$. Combining this fact with $OQ = OD$ and $CQ = AD$ implies that triangles ODA and OQC are congruent. Therefore $OA = OC$ and, if M is the midpoint of segment AC, it follows that OM is perpendicular to AC. Since $ABCD$ is a parallelogram, M is also the midpoint of DB. If K denotes the intersection of the line through P perpendicular to DP and the line through Q perpendicular to DQ, then K is diametrically opposite D on the circumcircle of DPQ and O is the midpoint of segment DK. This implies that OM is a midline of triangle DBK and hence that BK is parallel to OM which is perpendicular to AC. Therefore K lies on the altitude from B in triangle ABC, as desired. □

5. One hundred circles of radius one are positioned in the plane so that the area of any triangle formed by the centres of three of these circles is at most 2017. Prove that there is a line intersecting at least three of these circles.

   Solution: We will prove that given n circles, there is some line intersecting more than $\frac{n}{46}$ of them. Let S be the set of centers of the n circles. We will first show that there is a line ℓ such that the projections of the points in S lie in an interval of length at most $\sqrt{8068} < 90$ on ℓ. Let A and B be the pair of points in S that are farthest apart and let the distance between A and B be d. Now consider any point $C \in S$ distinct from A and B. The distance from C to the line AB must be at most $\frac{4034}{d}$ since triangle ABC has area at most 2017. Therefore if ℓ is a line perpendicular to AB, then the projections of S onto ℓ lie in an interval of length $\frac{8068}{d}$ centered at the intersection of ℓ and AB. Furthermore, all of these projections must lie on an interval of length at most d on ℓ since the largest distance between two of these projections is at most d. Since $\min(d, 8068/d) \leq \sqrt{8068} < 90$, this proves the claim.

   Now note that the projections of the n circles onto the line ℓ are intervals of length 2, all contained in an interval of length at most $\sqrt{8068} + 2 < 92$. Each point of this interval belongs to on average $\frac{2n}{\sqrt{8068}+2} > \frac{n}{46}$ of the subintervals of length 2 corresponding to the projections of the n circles onto ℓ. Thus there is some point $x \in \ell$ belonging to the projections of more than $\frac{n}{46}$ circles. The line perpendicular to ℓ through x has the desired property. Setting $n = 100$ yields that there is a line intersecting at least three of the circles. □

