Solutions to 2015 CMO (DRAFT—as of April 6, 2015)

Problem 1. *Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. Find all functions f, defined on $\mathbb{N}$ and taking values in $\mathbb{N}$, such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every positive integer n.*

Solution. The only such function is $f(n) = n$.

Assume that f satisfies the given condition. It will be shown by induction that $f(n) = n$ for all $n \in \mathbb{N}$. Substituting $n = 1$ yields that $0 < f(1)f(f(1)) < 2$ which implies the base case $f(1) = 1$. Now assume that $f(k) = k$ for all $k < n$ and assume for contradiction that $f(n) \neq n$.

On the one hand, if $f(n) \leq n-1$ then $f(f(n)) = f(n)$ and $f(n)f(f(n)) = f(n)^2 \leq (n-1)^2$ which is a contradiction. On the other hand, if $f(n) \geq n+1$ then there are several ways to proceed.

Method 1: Assume $f(n) = M \geq n+1$. Then $(n+1)f(M) \leq f(n)f(f(n)) < n^2 + n$. Therefore $f(M) < n$, and hence $f(f(M)) = f(M)$ and $f(M)f(f(M)) = f(M)^2 < n^2 \leq (M-1)^2$, which is a contradiction. This completes the induction. □

Method 2: First note that if $|a-b| > 1$, then the intervals $((a-1)^2, a^2+a)$ and $((b-1)^2, b^2+b)$ are disjoint which implies that $f(a)$ and $f(b)$ cannot be equal.

Assuming $f(n) \geq n+1$, it follows that $f(f(n)) < \frac{n^2+n}{f(n)} \leq n$. This implies that for some $a \leq n-1$, $f(a) = f(f(n))$ which is a contradiction since $|f(n) - a| \geq n + 1 - a \geq 2$. This completes the induction. □

Method 3: Assuming $f(n) \geq n+1$, it follows that $f(f(n)) < \frac{n^2+n}{f(n)} \leq n$ and $f(f(f(n))) = f(f(n))$. This implies that $(f(n)-1)^2 < f(f(n))f(f(f(n))) = f(f(n))^2 < f(n)^2 + f(n)$ and therefore that $f(f(n)) = f(n)$ since $f(n)^2$ is the unique square satisfying this constraint. This implies that $f(n)f(f(n)) = f(n)^2 \geq (n+1)^2$ which is a contradiction, completing the induction. □

Problem 2. *Let ABC be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that*

$$\frac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2\,.$$

Solution. Method 1: Let $AB = c$, $AC = b$, and $BC = a$ denote the three side lengths of the triangle.

As $\angle BFH = \angle BDH = 90^\circ$, $FHDB$ is a cyclic quadrilateral. By the Power-of-a-Point Theorem, $AH \cdot AD = AF \cdot AB$. (We can derive this result in other ways: for example, see Method 2, below.)

Since $AF = AC \cdot \cos\angle A$, we have $AH \cdot AD = AC \cdot AB \cdot \cos\angle A = bc\cos\angle A$.

By the Cosine Law, $\cos\angle A = \dfrac{b^2+c^2-a^2}{2bc}$, which implies that $AH \cdot AD = \dfrac{b^2+c^2-a^2}{2}$.

By symmetry, we can show that $BH \cdot BE = \dfrac{a^2+c^2-b^2}{2}$ and $CH \cdot CF = \dfrac{a^2+b^2-c^2}{2}$. Hence,

$$\begin{aligned} AH \cdot AD + BH \cdot BE + CH \cdot CF &= \frac{b^2+c^2-a^2}{2} + \frac{a^2+c^2-b^2}{2} + \frac{a^2+b^2-c^2}{2} \\ &= \frac{a^2+b^2+c^2}{2}. \end{aligned} \tag{1}$$

Our desired inequality, $\dfrac{AB \cdot AC + BC \cdot BA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \le 2$, is equivalent to the inequality $\dfrac{cb+ac+ba}{\frac{a^2+b^2+c^2}{2}} \le 2$, which simplifies to $2a^2+2b^2+2c^2 \ge 2ab+2bc+2ca$.

But this last inequality is easy to prove, as it is equivalent to $(a-b)^2+(a-c)^2+(b-c)^2 \ge 0$.

Therefore, we have established the desired inequality. The proof also shows that equality occurs if and only if $a=b=c$, i.e., $\triangle ABC$ is equilateral. □

Method 2: Observe that

$$\frac{AE}{AH} = \cos(\angle HAE) = \frac{AD}{AC} \quad \text{and} \quad \frac{AF}{AH} = \cos(\angle HAF) = \frac{AD}{AB}.$$

It follows that

$$AC \cdot AE = AH \cdot AD = AB \cdot AF\,.$$

By symmetry, we similarly have

$$BC \cdot BD = BH \cdot BE = BF \cdot BA \quad \text{and} \quad CD \cdot CB = CH \cdot CF = CE \cdot CA\,.$$

Therefore

$$\begin{aligned} &2(AH \cdot AD + BH \cdot BE + CH \cdot CF) \\ &= AB(AF+BF) + AC(AE+CE) + BC(BD+CD) \\ &= AB^2 + AC^2 + BC^2. \end{aligned}$$

This proves Equation (1) in Method 1. The rest of the proof is the same as the part of the proof of Method 1 that follows Equation (1). □

Problem 3. *On a $(4n+2)\times(4n+2)$ square grid, a turtle can move between squares sharing a side. The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?*

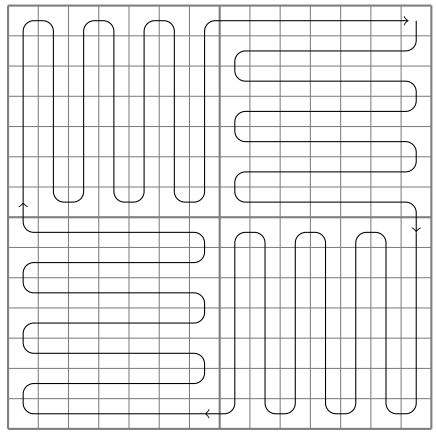
Solution. We shall prove that the answer is $2n+2$. Number the rows in increasing order, from top to bottom, and number the columns from left to right. By symmetry, we may (and shall) assume that the turtle starts in the top right corner square.

First we shall prove that some row or column must be entered at least $2n+2$ times. Let $m = 4n+2$. First note that each time the turtle moves, she enters either a row or a column. Let r_i denote the number of times the turtle enters row i, and let c_i be similarly defined for column i. Since the turtle moves m^2 times,

$$r_1 + r_2 + \cdots + r_m + c_1 + c_2 + \cdots + c_m = m^2.$$

Now note that each time the turtle enters column 1, the next column she enters must be column 2. Therefore c_1 is equal to the number of times the turtle enters column 2 from column 1. Furthermore, the turtle must enter column 2 from column 3 at least once, which implies that $c_2 > c_1$. Therefore since the $2m$ terms r_i and c_i are not all equal, one must be strictly greater than $m^2/(2m) = 2n+1$ and therefore at least $2n+2$.

Now we construct an example to show that it is possible that no row or column is entered more than $2n+2$ times. Partition the square grid into four $(2n+1) \times (2n+1)$ quadrants A, B, C, and D, containing the upper left, upper right, lower left, and lower right corners, respectively. The turtle begins at the top right corner square of B, moves one square down, and then moves left through the whole second row of B. She then moves one square down and moves right through the whole third row of B. She continues in this pattern, moving through each remaining row of B in succession and moving one square down when each row is completed. Since $2n+1$ is odd, the turtle ends at the bottom right corner of B. She then moves one square down into D and through each column of D in turn, moving one square to the left when each column is completed. She ends at the lower left corner of D and moves left into C and through the rows of C, moving one square up when each row is completed, ending in the upper left corner of C. She then enters A and moves through the columns of A, moving one square right when each column is completed. This takes her to the upper right corner of A, whereupon she enters B and moves right through the top row of B, which returns her to her starting point. Each row passing through A and B is entered at most $2n+1$ times in A and once in B, and thus at most $2n+2$ times in total. Similarly, each row and column in the grid is entered at most $2n+2$ times by this path. (See figure below.)



Problem 3: the case $n = 3$

$\square$

Problem 4. *Let ABC be an acute-angled triangle with circumcenter O. Let Γ be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that $BP \cdot CQ = AP \cdot AQ$. Prove that Γ is tangent to the circumcircle of triangle BOC.*

Solution. Let ω be the circumcircle of BOC. Let M be the point diametrically opposite to O on ω and let the line AM intersect ω at M and K. Since O is the circumcenter of ABC, it follows that $OB = OC$ and therefore that O is the midpoint of the arc $\widehat{BOC}$ of ω. Since M is diametrically opposite to O, it follows that M is the midpoint of the arc $\widehat{BMC}$ of ω. This implies since K is on ω that KM is the bisector of $\angle BKC$.Since K is on ω, this implies that $\angle BKM = \angle CKM$, i.e. KM is the bisector of $\angle BKC$.

Since O is the circumcenter of ABC, it follows that $\angle BOC = 2\angle BAC$. Since B, K, O and C all lie on ω, it also follows that $\angle BKC = \angle BOC = 2\angle BAC$. Since KM bisects $\angle BKC$, it follows that $\angle BKM = \angle CKM = \angle BAC$. The fact that A, K and M lie on a line therefore implies that $\angle AKB = \angle AKC = 180° - \angle BAC$. Now it follows that

$$\angle KBA = 180° - \angle AKB - \angle KAB = \angle BAC - \angle KAB = \angle KAC.$$

This implies that triangles KBA and KAC are similar. Rearranging the condition in the problem statement yields that $BP/AP = AQ/CQ$ which, when combined with the fact that KBA and KAC are similar, implies that triangles KPA and KQC are similar. Therefore $\angle KPA = \angle KQC = 180° - \angle KQA$ which implies that K lies on Γ.

Now let S denote the centre of Γ and let T denote the centre of ω. Note that T is the midpoint of segment OM and that TM and AS, which are both perpendicular to BC, are parallel. This implies that $\angle KMT = \angle KAS$ since A, K and M are collinear. Further, since KTM and KSA are isosceles triangles, it follows that $\angle TKM = \angle KMT$ and $\angle SKA =$

$\angle KSA$. Therefore $\angle TKM = \angle SKA$ which implies that S, T and K are collinear. Therefore Γ and ω intersect at a point K which lies on the line ST connecting the centres of the two circles. This implies that the circles Γ and ω are tangent at K. $\square$

Problem 5. *Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.*

Solution. First suppose $b \equiv \pm a \pmod{p}$ and $c \equiv \pm b \pmod{p}$. Then, for any n, we have $a^n + b^n + c^n \equiv \pm a^n$ or $\pm 3a^n \pmod{p}$. We are given that $p \neq 3$ (since $\frac{3-1}{2}$ is not prime) and $p \nmid a$, so it follows that $a^n + b^n + c^n \not\equiv 0 \pmod{p}$. The claim is trivial in this case. Otherwise, we may assume without loss of generality that $b \not\equiv \pm a \pmod{p} \implies ba^{-1} \not\equiv \pm 1 \pmod{p}$.

Now let $q = \frac{p-1}{2}$. By Fermat's little theorem, we know that the order of ba^{-1} mod p divides $p - 1 = 2q$. However, since $ba^{-1} \not\equiv \pm 1 \pmod{p}$, the order of ba^{-1} does not divide 2. Thus, the order must be either q or $2q$.

Next, let S denote the set of positive integers $n < p$ such that $a^n + b^n + c^n \equiv 0 \pmod{p}$, and let s_t denote the number of ordered pairs $(i, j) \subset S$ such that $i - j \equiv t \pmod{p-1}$.

Lemma: If t is a positive integer less than $2q$ and not equal to q, then $s_t \leq 2$.

Proof: Consider $i, j \in S$ with $j - i \equiv t \pmod{p-1}$. Then we have

$$\begin{aligned} & a^i + b^i + c^i \equiv 0 \pmod{p} \\ \implies \quad & a^i c^{j-i} + b^i c^{j-i} + c^j \equiv 0 \pmod{p} \\ \implies \quad & a^i c^{j-i} + b^i c^{j-i} - a^j - b^j \equiv 0 \pmod{p} \\ \implies \quad & a^i \cdot \left(c^t - a^t\right) \equiv b^i \cdot \left(b^t - c^t\right) \pmod{p}. \end{aligned}$$

If $c^t \equiv a^t \pmod{p}$, then this implies $c^t \equiv b^t \pmod{p}$ as well, so $\left(ab^{-1}\right)^t \equiv 1 \pmod{p}$. However, we know the order of ab^{-1} is q or $2q$, and $q \nmid t$, so this impossible. Thus, we can write

$$(ab^{-1})^i \equiv \left(b^t - c^t\right) \cdot \left(c^t - a^t\right)^{-1} \pmod{p}.$$

For a fixed t, the right-hand side of this equation is fixed, so $(ab^{-1})^i$ is also fixed. Since the order of ab^{-1} is either q or $2q$, it follows that there are at most 2 solutions for i, and the lemma is proven. $\square$

Now, for each element i in S, there are at least $|S| - 2$ other elements that differ from i by a quantity other than $q \pmod{p-1}$. Therefore, the lemma implies that

$$\begin{aligned} |S| \cdot (|S| - 2) \ &\leq \ \sum_{t \neq q} s_t \ \leq \ 2 \cdot (p-2) \\ \implies \quad (|S| - 1)^2 \ &\leq \ 2p - 3 \\ \implies \quad |S| \ &< \ \sqrt{2p} + 1. \end{aligned}$$

$\square$