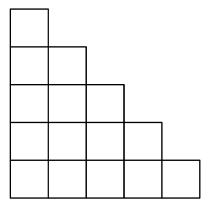
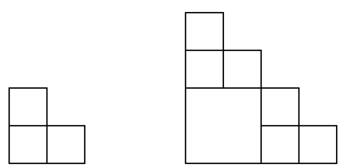
CANADIAN MATHEMATICAL OLYMPIAD 2010
PROBLEMS AND SOLUTIONS

(1) For a positive integer n, an *n-staircase* is a figure consisting of unit squares, with one square in the first row, two squares in the second row, and so on, up to n squares in the n^{th} row, such that all the left-most squares in each row are aligned vertically. For example, the 5-staircase is shown below.



Let $f(n)$ denote the minimum number of square tiles required to tile the n-staircase, where the side lengths of the square tiles can be any positive integer. For example, $f(2) = 3$ and $f(4) = 7$.



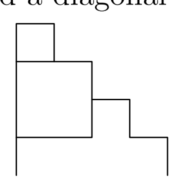
(a) Find all n such that $f(n) = n$.
(b) Find all n such that $f(n) = n + 1$.

Solution. (a) A *diagonal* square in an n-staircase is a unit square that lies on the diagonal going from the top-left to the bottom-right. A *minimal tiling* of an n-staircase is a tiling consisting of $f(n)$ square tiles.

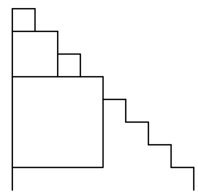
Observe that $f(n) \geq n$ for all n. There are n diagonal squares in an n-staircase, and a square tile can cover at most one diagonal square, so any tiling requires at least n square tiles. In other words, $f(n) \geq n$. Hence, if $f(n) = n$, then each square tile covers exactly one diagonal square.

Let n be a positive integer such that $f(n) = n$, and consider a minimal tiling of an n-staircase. The only square tile that can cover the unit square in the first row is the unit square itself.

Now consider the left-most unit square in the second row. The only square tile that can cover this unit square and a diagonal square is a 2×2 square tile.



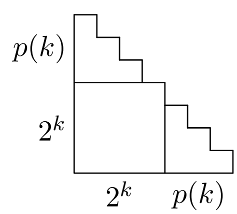
Next, consider the left-most unit square in the fourth row. The only square tile that can cover this unit square and a diagonal square is a 4×4 square tile.



Continuing this construction, we see that the side lengths of the square tiles we encounter will be 1, 2, 4, and so on, up to 2^k for some nonnegative integer k. Therefore, n, the height of the n-staircase, is equal to $1+2+4+\cdots+2^k = 2^{k+1}-1$. Alternatively, $n = 2^k - 1$ for some positive integer k. Let $p(k) = 2^k - 1$.

Conversely, we can tile a $p(k)$-staircase with $p(k)$ square tiles recursively as follows: We have that $p(1) = 1$, and we can tile a 1-staircase with 1 square tile. Assume that we can tile a $p(k)$-staircase with $p(k)$ square tiles for some positive integer k.

Consider a $p(k+1)$-staircase. Place a $2^k \times 2^k$ square tile in the bottom left corner. Note that this square tile covers a digaonal square. Then $p(k+1) - 2^k = 2^{k+1} - 1 - 2^k = 2^k - 1 = p(k)$, so we are left with two $p(k)$-staircases.

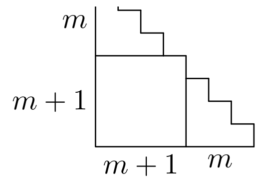


Furthermore, these two $p(k)$-staircases can be tiled with $2p(k)$ square tiles, which means we use $2p(k) + 1 = p(k+1)$ square tiles.

Therefore, $f(n) = n$ if and only if $n = 2^k - 1 = p(k)$ for some positive integer k. In other words, the binary representation of n consists of all 1s, with no 0s.

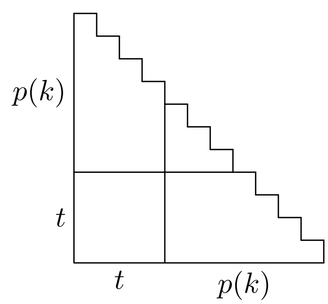
(b) Let n be a positive integer such that $f(n) = n + 1$, and consider a minimal tiling of an n-staircase. Since there are n diagonal squares, every square tile except one covers a diagonal square. We claim that the square tile that covers the bottom-left unit square must be the square tile that does not cover a diagonal square.

If n is even, then this fact is obvious, because the square tile that covers the bottom-left unit square cannot cover any diagonal square, so assume that n is odd. Let $n = 2m + 1$. We may assume that $n > 1$, so $m \geq 1$. Suppose that the square tile covering the bottom-left unit square also covers a diagonal square. Then the side length of this square tile must be $m + 1$. After this $(m+1) \times (m+1)$ square tile has been placed, we are left with two m-staircases.

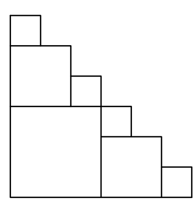


Hence, $f(n) = 2f(m) + 1$. But $2f(m) + 1$ is odd, and $n + 1 = 2m + 2$ is even, so $f(n)$ cannot be equal to $n + 1$, contradiction. Therefore, the square tile that covers the bottom-left unit square is the square tile that does not cover a diagonal square.

Let t be the side length of the square tile covering the bottom-left unit square. Then every other square tile must cover a diagonal square, so by the same construction as in part (a), $n = 1+2+4+\cdots+2^{k-1}+t = 2^k+t-1$ for some positive integer k. Furthermore, the top $p(k) = 2^k - 1$ rows of the n-staircase must be tiled the same way as the minimal tiling of a $p(k)$-staircase. Therefore, the horizontal line between rows $p(k)$ and $p(k) + 1$ does not pass through any square tiles. Let us call such a line a *fault line*. Similarly, the vertical line between columns t and $t + 1$ is also a fault line. These two fault lines partition two $p(k)$-staircases.



If these two $p(k)$-staircases do not overlap, then $t = p(k)$, so $n = 2p(k)$. For example, the minimal tiling for $n = 2p(2) = 6$ is shown below.



Hence, assume that the two $p(k)$-staircases do overlap. The intersection of the two $p(k)$-staircases is a $[p(k) - t]$-staircase. Since this $[p(k) - t]$-staircase is tiled the same way as the top $p(k) - t$ rows of a minimal tiling of a $p(k)$-staircase, $p(k) - t = p(l)$ for some positive integer $l < k$, so $t = p(k) - p(l)$. Then

$$n = t + p(k) = 2p(k) - p(l).$$

Since $p(0) = 0$, we can summarize by saying that n must be of the form

$$n = 2p(k) - p(l) = 2^{k+1} - 2^l - 1,$$

where k is a positive integer and l is a nonnegative integer. Also, our argument shows how if n is of this form, then an n-staircase can be tiled with $n+1$ square tiles.

Finally, we observe that n is of this form if and only if the binary representation of n contains exactly one 0:

$$2^{k+1} - 2^l - 1 = \underbrace{11\ldots1}_{k-l \text{ 1s}}0\underbrace{11\ldots1}_{l \text{ 1s}}.$$

□

(2) Let A, B, P be three points on a circle. Prove that if a and b are the distances from P to the tangents at A and B and c is the distance from P to the chord AB, then $c^2 = ab$.

Solution. Let r be the radius of the circle, and let a' and b' be the respective lengths of PA and PB. Since $b' = 2r \sin \angle PAB = 2rc/a'$, $c = a'b'/(2r)$. Let AC be the diameter of the circle and H the foot of the perpendicular from P to AC. The similarity of the triangles ACP and APH imply that $AH : AP = AP : AC$ or $(a')^2 = 2ra$. Similarly, $(b')^2 = 2rb$. Hence

$$c^2 = \frac{(a')^2}{2r}\frac{(b')^2}{2r} = ab$$

as desired. □

Alternate Solution. Let E, F, G be the feet of the perpendiculars to the tangents at A and B and the chord AB, respectively. We need to show that $PE : PG = PG : GF$, where G is the foot of the perpendicular from P to AB. This suggest that we try to prove that the triangles EPG and GPF are similar.

Since PG is parallel to the bisector of the angle between the two tangents, $\angle EPG = \angle FPG$. Since $AEPG$ and $BFPG$ are concyclic quadrilaterals (having opposite angles right), $\angle PGE = \angle PAE$ and $\angle PFG = \angle PBG$. But $\angle PAE = \angle PBA = \angle PBG$, whence $\angle PGE = \angle PFG$. Therefore triangles EPG and GPF are similar.

The argument above with concyclic quadrilaterals only works when P lies on the shorter arc between A and B. The other case can be proved similarly. □

(3) Three speed skaters have a friendly race on a skating oval. They all start from the same point and skate in the same direction, but with different speeds that they maintain throughout the race. The slowest skater does 1 lap a minute, the fastest one does 3.14 laps a minute, and the middle one does L laps a minute for some $1 < L < 3.14$. The race ends at the moment when all three skaters again come together to the same point on the oval (which may differ from the starting

point.) Find how many different choices for L are there such that 117 passings occur before the end of the race. (A passing is defined when one skater passes another one. The beginning and the end of the race when all three skaters are at together are not counted as a passing.)

Solution. Assume that the length of the oval is one unit. Let $x(t)$ be the difference of distances that the slowest and the fastest skaters have skated by time t. Similarly, let $y(t)$ be the difference between the middle skater and the slowest skater. The path $(x(t), y(t))$ is a straight ray R in $\mathbb{R}^2$, starting from the origin, with slope depending on L. By assumption, $0 < y(t) < x(t)$.

One skater passes another one when either $x(t) \in \mathbb{Z}$, $y(t) \in \mathbb{Z}$ or $x(t) - y(t) \in \mathbb{Z}$. The race ends when both $x(t), y(t) \in \mathbb{Z}$.

Let $(a, b) \in \mathbb{Z}^2$ be the endpoint of the ray R. We need to find the number of such points satisfying:

(a) $0 < b < a$

(b) The ray R intersects $\mathbb{Z}^2$ at endpoints only.

(c) The ray R crosses 357 times the lines $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, $y - x \in \mathbb{Z}$.

The second condition says that a and b are relatively prime. The ray R crosses $a-1$ of the lines $x \in \mathbb{Z}$, $b-1$ of the lines $y \in \mathbb{Z}$ and $a-b-1$ of the lines $x - y \in \mathbb{Z}$. Thus, we need $(a-1) + (b-1) + (a-b-1) = 117$, or equivalently, $2a - 3 = 117$. That is $a = 60$.

Now b must be a positive integer less than and relatively prime to 60. The number of such b can be found using the Euler's ϕ function:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2-1) \cdot 2 \cdot (3-1) \cdot (5-1) = 16.$$

Thus the answer is 16. □

Alternate Solution. First, let us name our skaters. From fastest to slowest, call them: A, B and C. (Abel, Bernoulli and Cayley?)

Now, it is helpful to consider the race from the viewpoint of C. Relative to C, both A and B complete a whole number of laps, since they both start and finish at C.

Let n be the number of laps completed by A relative to C, and let m be the number of laps completed by B relative to C. Note that: $n > m \in \mathbb{Z}^+$

Consider the number of minutes required to complete the race. Relative to C, A is moving with a speed of $3.14 - 1 = 2.14$ laps per minute and completes the race in $\frac{n}{2.14}$ minutes. Also relative to C, B is moving with a speed of $(L-1)$ laps per minute and completes the race in $\frac{m}{L-1}$ minutes. Since A and B finish the race together (when they both meet C):

$$\frac{n}{2.14} = \frac{m}{L-1} \qquad \Rightarrow \qquad L = 2.14\left(\frac{m}{n}\right) + 1.$$

Hence, there is a one-to-one relation between values of L and values of the postive proper fraction $\frac{m}{n}$. The fraction should be reduced, that is the pair (m, n) should be relatively prime, or else, with $k = \gcd(m, n)$, the race ends after n/k laps for A and m/k laps for B when they *first* meet C together.

It is also helpful to consider the race from the viewpoint of B. In this frame of reference, A completes only $n - m$ laps. Hence A *passes* B only $(n - m) - 1$ times, since the racers do not "pass" at the end of the race (nor at the beginning). Similarily A passes C only $n - 1$ times and B passes C only $m - 1$ times. The total number of passings is:

$$117 = (n - 1) + (m - 1) + (n - m - 1) = 2n - 3 \quad \Rightarrow \quad n = 60$$

Hence the number of values of L equals the number of m for which the fraction $\frac{m}{60}$ is positive, proper and reduced. That is the number of positive integer values smaller than and relatively prime to 60. One could simply count: {1,7,11,13,17,...}, but Euler's ϕ function gives this number:

$$\phi(60) = \phi(2^2 \cdot 3 \cdot 5) = (2 - 1) \cdot 2 \cdot (3 - 1) \cdot (5 - 1) = 16.$$

Therefore, there are 16 values for L which give the desired number of passings.

Note that the actual values for the speeds of A and C do not affect the result. They could be any values, rational or irrational, just so long as they are different, and there will be 16 possible values for the speed of B between them. □

(4) Each vertex of a finite graph can be colored either black or white. Initially all vertices are black. We are allowed to pick a vertex P and change the color of P and all of its neighbours. Is it possible to change the colour of every vertex from black to white by a sequence of operations of this type?

Solution. The answer is yes. Proof by induction on the number n of vertices. If $n = 1$, this is obvious. For the induction assumption, suppose we can do this for any graph with $n-1$ vertices for some $n \geq 2$ and let X be a graph with n vertices which we will denote by $P_1, \dots, P_{n+1}$.

Let us denote the "basic" operation of changing the color of P_i and all of its neighbours by f_i. Removing a vertex P_i from X (along with all edges connecting to P_i) and applying the induction assumption to the resulting smaller graph, we see that there exists a sequence of operations g_i (obtained by composing some f_j, with $j \neq i$) which changes the colour of every vertex in X, except for possibly P_i.

If g_i it also changes the color of P_i then we are done. So, we may assume that g_i does not change the colour of P for every $i = 1, \dots, n$. Now consider two cases.

Case 1: n is even. Then composing $g_1, \dots, g_n$ we will change the color of every vertex from white to black.

Case 2: n is odd. I claim that in this case X has a vertex with an even number of neighbours.

Indeed, denote the number of neighbours of P_i (or equivalently, the number of edges connected to P) by k_i. Then $P_1 + \dots + P_{n+1} = 2e$, where e is the number of edges of X. Thus one of the numbers k_i has to be even as claimed.

After renumbering the vertices, we may assume that P_1 has $2k$ neighbours, say $P_2, \dots, P_{2k+1}$. The composition of f_1 with $g_1, g_2, \dots, g_{2k+1}$ will then change the colour of every vertex, as desired.

□

(5) Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Let $a_n = n! + n$. Show that if $P(a_n)/Q(a_n)$ is an integer for every n, then $P(n)/Q(n)$ is an integer for every integer n such that $Q(n) \neq 0$.

Solution. Imagine dividing $P(x)$ by $Q(x)$. We find that

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)},$$

where $A(x)$ and $R(x)$ are polynomials with rational coefficients, and $R(x)$ is either identically 0 or has degree less than the degree of $Q(x)$.

By bringing the coefficients of $A(x)$ to their least common multiple, we can find a polynomial $B(x)$ with integer coefficients, and a positive integer b, such that $A(x) = B(x)/b$. Suppose first that $R(x)$ is not identically 0. Note that for any integer k, either $A(k) = 0$, or $|A(k)| \geq 1/b$. But whenever $|k|$ is large enough, $0 < |R(k)/Q(k)| < 1/b$, and therefore if n is large enough, $P(a_n)/Q(a_n)$ cannot be an integer.

So $R(x)$ is identically 0, and $P(x)/Q(x) = B(x)/b$ (at least whenever $Q(x) \neq 0$.)

Now let n be an integer. Then there are infinitely many integers k such that $n \equiv a_k \pmod{b}$. But $B(a_k)/b$ is an integer, or equivalently b divides $B(a_k)$. It follows that b divides $B(n)$, and therefore $P(n)/Q(n)$ is an integer. □