1997
SOLUTIONS

Problem 1 – Deepee Khosla, Lisgar Collegiate Institute, Ottawa, ON

Let $p_1, \ldots, p_{12}$ denote, in increasing order, the primes from 7 to 47. Then

$$5! = 2^3 \cdot 3^1 \cdot 5^1 \cdot p_1^0 \cdot p_2^0 \ldots p_{12}^0$$

and

$$50! = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot p_1^{b_1} \cdot p_2^{b_2} \ldots p_{12}^{b_{12}}.$$

Note that $2^4, 3^2, 5^2, p_1, \ldots, p_{12}$ all divide 50!, so all its prime powers differ from those of 5!

Since $x, y | 50!$, they are of the form

$$\begin{aligned} x &= 2^{n_1} \cdot 3^{n_2} \cdot \ldots p_{12}^{n_{15}} \\ y &= 2^{m_1} \cdot 3^{m_2} \cdot \ldots p_{12}^{m_{15}}. \end{aligned}$$

Then $\max(n_i, m_i)$ is the ith prime power in 50!
and $\min(n_i, m_i)$ is the ith prime power in 5!

Since, by the above note, the prime powers for p_{12} and under differ in 5! and 50!, there are 2^{15} choices for x, only half of which will be less than y. (Since for each choice of x, y is forced and either $x < y$ or $y < x$.) So the number of pairs is $2^{15}/2 = 2^{14}$.

Problem 2 – Byung Kuy Chun, Harry Ainlay Composite High School, Edmonton, AB

Look at the first point of each given unit interval. This point uniquely defines the given unit interval.

Lemma. In any interval $[x, x+1)$ there must be at least one of these first points ($0 \leq x \leq 49$).

Proof. Suppose the opposite. The last first point before x must be $x - \varepsilon$ for some $\varepsilon > 0$. The corresponding unit interval ends at $x - \varepsilon + 1 < x + 1$. However, the next given unit interval cannot begin until at least $x + 1$.

This implies that points $(x - \varepsilon + 1,\ x + 1)$ are not in set A, a contradiction.

$\therefore$ There must be a first point in $[x,\ x+1)$. $\square$

Note that for two first points in intervals $[x, x+1)$ and $[x+2, x+3)$ respectively, the corresponding unit intervals are disjoint since the intervals are in the range $[x, x+2)$ and $[x+2, x+4)$ respectively.

$\therefore$ We can choose a given unit interval that begins in each of

$$[0,1)[2,3)\ldots[2k,2k+1)\ldots[48,49).$$

Since there are 25 of these intervals, we can find 25 points which correspond to 25 disjoint unit intervals.

Problem 2 – Colin Percival, Burnaby Central Secondary School, Burnaby, BC

I prove the more general result, that if $[0, 2n] = \bigcup_i A_i$, $|A_i| = 1$, A_i are intervals then $\exists a_1 \ldots a_n$, such that $A_{a_i} \bigcap A_{a_j} = \emptyset$.

Let $0 < \varepsilon \leq \dfrac{2}{n-1}$ and let $b_i = (i-1)(2+\varepsilon), i = 1 \ldots n$. Then

$$\min\{b_i\} = 0, \max\{b_i\} = (n-1)(2+\varepsilon) \leq (n-1)\left(2 + \frac{2}{n-1}\right) = (n-1)\left(\frac{2n}{n-1}\right) = 2n.$$

So all the b_i are in $[0, 2n]$.

Let a_i be such that $b_i \in A_{a_i}$. Since $\bigcup A_i = [0, 2n]$, this is possible.

Then since $(b_i - b_j) = (i-j)(2+\varepsilon) \geq 2 + \varepsilon > 2$, and the A_i are intervals of length 1, $\min A_{a_i} - \max A_{a_j} > 2 - 1 - 1 = 0$, so $A_{a_i} \bigcap A_{a_j} = \emptyset$.

Substituting $n = 25$, we get the required result. Q.E.D.

Problem 3 – Mihaela Enachescu, Dawson College, Montréal, PQ

Let $P = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{1997}{1998}$. Then $\frac{1}{2} > \frac{1}{3}$ because $2 < 3$, $\frac{3}{4} > \frac{3}{5}$ because $4 < 5, \ldots,$

$\ldots \frac{1997}{1998} > \frac{1997}{1999}$ because $1998 < 1999$.

So

$$P > \frac{1}{3} \cdot \frac{3}{5} \cdot \ldots \cdot \frac{1997}{1999} = \frac{1}{1999} \quad . \qquad (1)$$

Also $\frac{1}{2} < \frac{2}{3}$ because $1 \cdot 3 < 2 \cdot 2$, $\frac{3}{4} < \frac{4}{5}$ because $3 \cdot 5 < 4 \cdot 4, \ldots$

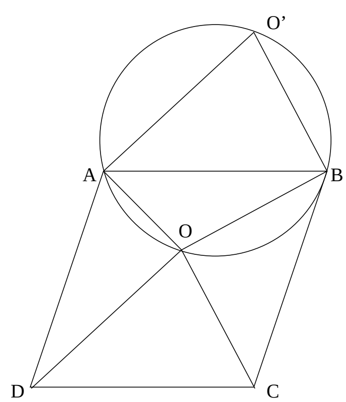
$\frac{1997}{1998} < \frac{1998}{1999}$ because $1997 \cdot 1999 = 1998^2 - 1 < 1998^2$.

So $P < \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{1998}{1999} = \underbrace{\left(\frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{1998}{1997}\right)}_{\frac{1}{P}} \frac{1}{1999}$.

$$\text{Hence } P^2 < \frac{1}{1999} < \frac{1}{1936} = \frac{1}{44^2} \text{ and } P < \frac{1}{44} \quad . \qquad (2)$$

Then (1) and (2) give $\frac{1}{1999} < P < \frac{1}{44}$ (q.e.d.)

Problem 4 – Joel Kamnitzer, Earl Haig Secondary School, North York, ON



Consider a translation which maps D to A. It will map $0 \to 0'$ with $\overline{OO'} = \overline{DA}$, and C will be mapped to B because $\overline{CB} = \overline{DA}$.

This translation keeps angles invariant, so $\angle AO'B = \angle DOC = 180° - \angle AOB$.

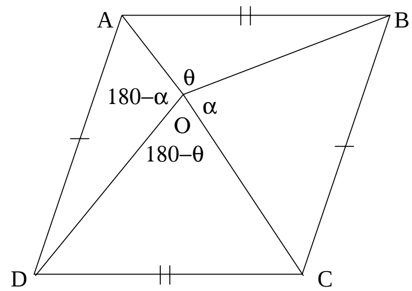
$\therefore AOBO'$ is a cyclic quadrilateral.

$\therefore \angle ODC = \angle O'AB = \angle O'OB$

but, since $O'O$ is parallel to BC,

$$\begin{aligned} \angle O'OB &= \angle OBC \\ \therefore \angle ODC &= \angle OBC. \end{aligned}$$

Problem 4 – Adrian Chan, Upper Canada College, Toronto, ON



Let $\angle AOB = \theta$ and $\angle BOC = \alpha$. Then $\angle COD = 180^\circ - \theta$ and $\angle AOD = 180^\circ - \alpha$.

Since $AB = CD$ (parallelogram) and $\sin\theta = \sin(180^\circ - \theta)$, the sine law on $\triangle OCD$ and $\triangle OAB$ gives

$$\frac{\sin\angle CDO}{OC} = \frac{\sin(180^\circ - \theta)}{CD} = \frac{\sin\theta}{AB} = \frac{\sin\angle ABO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ABO}{\sin\angle CDO}. \tag{1}$$

Similarily, the sine law on $\triangle OBC$ and $\triangle OAD$ gives

$$\frac{\sin\angle CBO}{OC} = \frac{\sin\alpha}{BC} = \frac{\sin(180^\circ - \alpha)}{AD} = \frac{\sin\angle ADO}{OA}$$

so

$$\frac{OA}{OC} = \frac{\sin\angle ADO}{\sin\angle CBO}. \tag{2}$$

Equations (1) and (2) show that $\sin\angle ABO \cdot \sin\angle CBO = \sin\angle ADO \cdot \sin\angle CDO$ hence

$$\frac{1}{2}[\cos(\angle ABO + \angle CBO) - \cos(\angle ABO - \angle CBO)] = \frac{1}{2}[\cos(\angle ADO + \angle CDO) - \cos(\angle ADO - \angle CDO)].$$

Since $\angle ADC = \angle ABC$(parallelogram) and $\angle ADO + \angle CDO = \angle ADC$ and $\angle ABO + \angle CBO = \angle ABC$ it follows that $\cos(\angle ABO - \angle CBO) = \cos(\angle ADO - \angle CDO)$.

There are two cases to consider.

Case (i): $\angle ABO - \angle CBO = \angle ADO - \angle CDO$.

Since $\angle ABO + \angle CBO = \angle ADO + \angle CDO$, subtracting gives $2\,\angle CBO = 2\,\angle CDO$ so $\angle CBO = \angle CDO$, and we are done.

Case (ii): $\angle ABO - \angle CBO = \angle CDO - \angle ADO$.

Since we know that $\angle ABO + \angle CBO = \angle CDO + \angle ADO$, adding gives $2\,\angle ABO = 2\,\angle CDO$ so $\angle ABO = \angle CDO$ and $\angle CBO = \angle ADO$.

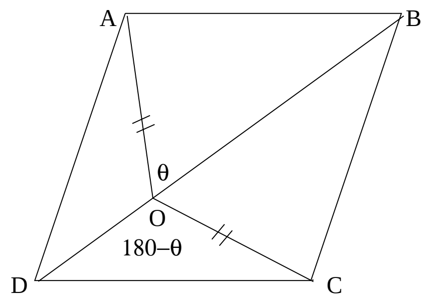
Substituting this into (1), it follows that $OA = OC$.

Also, $\angle ADO + \angle ABO = \angle CBO + \angle ABO = \angle ABC$.

Now, $\angle ABC = 180^\circ - \angle BAD$ since $ABCD$ is a parallelogram.

Hence $\angle BAD + \angle ADO + \angle ABO = 180^\circ$ so $\angle DOB = 180^\circ$ and D, O, B are collinear.

We now have the diagram



Then $\angle COD + \angle BOC = 180°$, so $\angle BOC = \theta = \angle AOB$.

$\triangle AOB$ is congruent to $\triangle COB$ (SAS, OB is common, $\angle AOB = \angle COB$ and $AO = CO$), so $\angle ABO = \angle CBO$. Since also $\angle ABO = \angle CDO$ we conclude that $\angle CBO = \angle CDO$.

Since it is true in both cases, then $\angle CBO = \angle CDO$.

Q.E.D.

Problem 5 – Sabin Cautis, Earl Haig Secondary School, North York, ON

We first note that

$$k^3 + 9k^2 + 26k + 24 = (k+2)(k+3)(k+4).$$

Let $S(n) = \displaystyle\sum_{k=0}^{n} \frac{(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}}{k^2 + 9k^2 + 26k + 24}$.

Then

$$\begin{aligned} S(n) &= \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!(k+2)(k+3)(k+4)} \\ &= \sum_{k=0}^{n} \left(\frac{(-1)^k (n+4)!}{(k+4)!(n-k)!} \right) \times \left(\frac{k+1}{(n+1)(n+2)(n+3)(n+4)} \right). \end{aligned}$$

Let

$$T(n) = (n+1)(n+2)(n+3)(n+4)S(n) = \sum_{k=0}^{n} \left((-1)^k \begin{pmatrix} n+4 \\ k+4 \end{pmatrix} (k+1) \right).$$

Now, for $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} = 0 \qquad (*)$$

since

$$(1-1)^n = \begin{pmatrix} n \\ 0 \end{pmatrix} - \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \ldots + (-1)^n \begin{pmatrix} n \\ n \end{pmatrix} = 0.$$

Also

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} i &= \sum_{i=1}^{n} (-1)^i \frac{i \cdot n!}{i! \cdot (n-i)!} + (-1)^0 \cdot \frac{0 \cdot n!}{0! \cdot n!} \\ &= \sum_{i=1}^{n} (-1)^i \frac{n!}{(i-1)!(n-i)!} \\ &= \sum_{i=1}^{n} (-1)^i n \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \\ &= n \sum_{i=1}^{n} (-1)^i \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} \\ &= -n \sum_{i=1}^{n} (-1)^{i-1} \begin{pmatrix} n-1 \\ i-1 \end{pmatrix}. \end{aligned}$$

Substituting $j = i - 1$, (*) shows that

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} i = -n\sum_{j=0}^{n-1}(-1)^j \binom{n-1}{j} = 0. \qquad (**)$$

Hence

$$\begin{aligned}
T(n) &= \sum_{k=0}^{n}(-1)^k \binom{n+4}{k+4}(k+1) \\
&= \sum_{k=0}^{n}(-1)^{k+4} \binom{n+4}{k+4}(k+1) \\
&= \sum_{k=-4}^{n}(-1)^{k+4} \binom{n+4}{k+4}(k+1) - \left(-3 + 2(n+4) - \binom{n+4}{2}\right).
\end{aligned}$$

Substituting $j = k + 4$,

$$\begin{aligned}
&= \sum_{j=0}^{n+4}(-1)^j \binom{n+4}{j}(j-3) - \left(2n + 8 - 3 - \frac{(n+4)(n+3)}{2}\right) \\
&= \sum_{j=0}^{n+4}(-1)^j \binom{n+4}{j} j - 3\sum_{j=0}^{n+4}(-1)^j \binom{n+4}{j} - \frac{1}{2}(4n + 10 - n^2 - 7n - 12)
\end{aligned}$$

The first two terms are zero because of results (*) and (**) so

$$T(n) = \frac{n^2 + 3n + 2}{2}.$$

Then

$$\begin{aligned}
S(n) &= \frac{T(n)}{(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{n^2 + 3n + 2}{2(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{(n+1)(n+2)}{2(n+1)(n+2)(n+3)(n+4)} \\
&= \frac{1}{2(n+3)(n+4)}.
\end{aligned}$$

$$\therefore \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24} = \frac{1}{2(n+3)(n+4)}$$