Canadian Junior Mathematical Olympiad 2020

Official Solutions

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1. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers that satisfies

$$a_1 = 1 \quad \text{and} \quad a_{n+1}^2 + a_{n+1} = a_n \quad \text{for every natural number } n.$$

Prove that $a_n \geq \frac{1}{n}$ for every natural number n.

Solution: We prove the inequality by induction. To start, observe that the inequality is obvious for $n = 1$.

Assume that the inequality $a_n \geq \frac{1}{n}$ holds for a given value of n. Let f be the function $f(x) = x^2 + x$, so that we have $f(a_{n+1}) = a_n$. In order to prove that $a_{n+1} \geq \frac{1}{n+1}$, it suffices to show that $f(a_{n+1}) \geq f\left(\frac{1}{n+1}\right)$ (since f is an increasing function on the positive real numbers). We have

$$\begin{aligned} f\left(\frac{1}{n+1}\right) &= \left(\frac{1}{n+1}\right)^2 + \frac{1}{n+1} \\ &= \frac{n+2}{(n+1)^2} \\ &= \frac{n^2+2n}{n(n+1)^2} \\ &< \frac{(n+1)^2}{n(n+1)^2} \\ &= \frac{1}{n} \\ &\leq a_n = f(a_{n+1}). \end{aligned}$$

Thus we can conclude that $a_{n+1} \geq \frac{1}{n+1}$.

This completes the induction proof.

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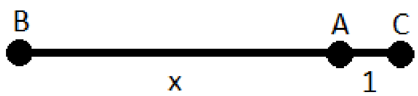
A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

2. Ziquan makes a drawing in the plane for art class. He starts by placing his pen at the origin, and draws a series of line segments, such that the n^{th} line segment has length n. He is not allowed to lift his pen, so that the end of the n^{th} segment is the start of the $(n+1)^{th}$ segment. Line segments drawn are allowed to intersect and even overlap previously drawn segments.

   After drawing a finite number of line segments, Ziquan stops and hands in his drawing to his art teacher. He passes the course if the drawing he hands in is an N by N square, for some positive integer N, and he fails the course otherwise. Is it possible for Ziquan to pass the course?

   Solution: We will prove that Ziquan can pass the course by drawing a square with side length $N = 54$. First, if Ziquan draws a line semgent of length x from point A to B, then he can draw a segment of length $x+1$ backwards from B to C, landing at the point C which is on the line AB and one unit right of A.



   This has the net effect of drawing a line segment of length 1 with a tail of length x in the opposite direction. Thus if Ziquan has already drawn the segment AB, the net effect is extending the existing line segment by 1 unit. We call this a "unit shift".

   Ziquan starts by drawing the first 11 line segments on the $x-$axis, all going to the right, except the segment of length 6 which goes to the left. This creates a line segment from $A = (0, 0)$ to $B = (54, 0)$, as $54 = 1 + 2 + 3 + 4 + 5 - 6 + 7 + 8 + 9 + 10 + 11$. Ziquan draws the second side of the square by drawing vertical segments going up of lengths $12, 13, 14, 15$, ending at $C = (54, 54)$. He goes left by $16, 17, 18$, which puts him at $(3, 54)$. Three unit shifts left lands him at $D = (0, 54)$, having just drawn a segment of length 24. He finishes the square by going down with segments of length $25, 26$, followed by three final unit shifts down. Note that every unit shift done has sufficient tail.

   Alternate Solution: Yes, for $N = 30$: Start in one direction with $1 + 2 + 3$, turn right for $4+5+6+7+8$, turn right for $9+10+11$, turn right for $12+13-14+15-16+17-18+19-20+21-22+23$, turn right for 24.

   Other Possible Solutions: For $N = 78$ $(n = 56)$, $N = 120$ (with two different number of steps n, 119 and 71), and $N = 190$ $(n = 149)$.

   □

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3. Let S be a set of $n \geq 3$ positive real numbers. Show that the largest possible number of distinct integer powers of three that can be written as the sum of three distinct elements of S is $n-2$.

   Solution: We will show by induction that for all $n \geq 3$, it holds that at most $n-2$ powers of three are sums of three distinct elements of S for any set S of positive real numbers with $|S| = n$. This is trivially true when $n = 3$. Let $n \geq 4$ and consider the largest element $x \in S$. The sum of x and any two other elements of S is strictly between x and $3x$. Therefore x can be used as a summand for at most one power of three. By the induction hypothesis, at most $n-3$ powers of three are sums of three distinct elements of $S\backslash\{x\}$. This completes the induction.

   Even if it was not asked to prove, we will now show that the optimal answer $n-2$ is reached. Observe that the set $S = \{1, 2, 3^2 - 3, 3^3 - 3, \ldots, 3^n - 3\}$ is such that $3^2, 3^3, \ldots, 3^n$ can be expressed as sums of three distinct elements of S. This makes use of the fact that each term of the form $3^k - 3$ can be used in exactly one sum of three terms equal to 3^k.

   $\square$

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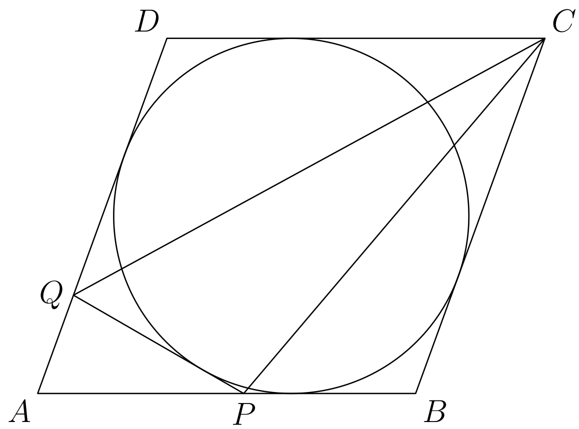
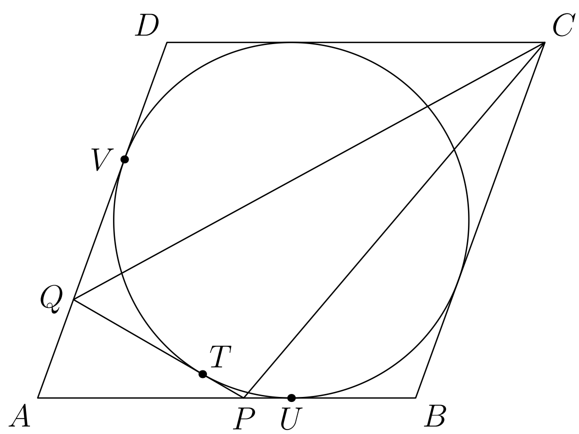
4. A circle is inscribed in a rhombus $ABCD$. Points P and Q vary on line segments $\overline{AB}$ and $\overline{AD}$, respectively, so that $\overline{PQ}$ is tangent to the circle. Show that for all such line segments $\overline{PQ}$, the area of triangle CPQ is constant.



Solution: Let the circle be tangent to $\overline{PQ}$, $\overline{AB}$, $\overline{AD}$ at T, U, and V, respectively. Let $p = PT = PU$ and $q = QT = QV$. Let $a = AU = AV$ and $b = BU = DV$. Then the side length of the rhombus is $a + b$.



Let $\theta = \angle BAD$, so $\angle ABC = \angle ADC = 180^\circ - \theta$. Then (using the notation [XYZ] for the area of a triangle of vertices X, Y, Z)

$$[APQ] = \frac{1}{2} \cdot AP \cdot AQ \cdot \sin\theta = \frac{1}{2}(a-p)(a-q)\sin\theta,$$
$$[BCP] = \frac{1}{2} \cdot BP \cdot BC \cdot \sin(180^\circ - \theta) = \frac{1}{2}(b+p)(a+b)\sin\theta,$$
$$[CDQ] = \frac{1}{2} \cdot DQ \cdot CD \cdot \sin(180^\circ - \theta) = \frac{1}{2}(b+q)(a+b)\sin\theta,$$

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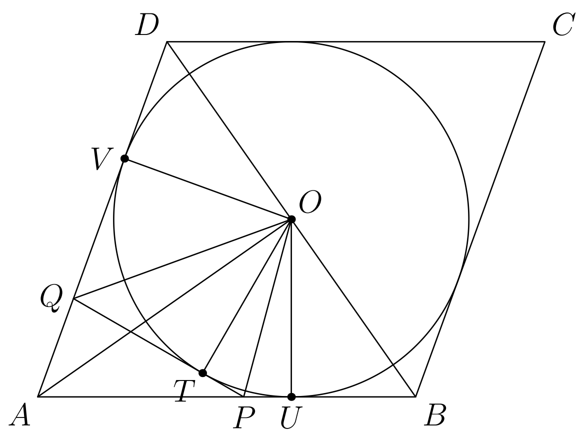
so

$$\begin{aligned}[CPQ] &= [ABCD] - [APQ] - [BCP] - [CDQ] \\ &= (a+b)^2 \sin\theta - \frac{1}{2}(a-p)(a-q)\sin\theta - \frac{1}{2}(b+p)(a+b)\sin\theta - \frac{1}{2}(b+q)(a+b)\sin\theta \\ &= \frac{1}{2}(a^2 + 2ab - bp - bq - pq)\sin\theta.\end{aligned}$$

Let O be the center of the circle, and let r be the radius of the circle. Let $x = \angle TOP = \angle UOP$ and $y = \angle TOQ = \angle VOQ$. Then $\tan x = \frac{p}{r}$ and $\tan y = \frac{q}{r}$.



Note that $\angle UOV = 2x + 2y$, so $\angle AOU = x + y$. Also, $\angle AOB = 90°$, so $\angle OBU = x + y$. Therefore,

$$\tan(x+y) = \frac{a}{r} = \frac{r}{b},$$

so $r^2 = ab$. But

$$\frac{r}{b} = \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{p}{r} + \frac{q}{r}}{1 - \frac{p}{r}\cdot\frac{q}{r}} = \frac{r(p+q)}{r^2 - pq} = \frac{r(p+q)}{ab - pq}.$$

Hence, $ab - pq = bp + bq$, so $bp + bq + pq = ab$. Therefore,

$$[CPQ] = \frac{1}{2}(a^2 + 2ab - bp - bq - pq)\sin\theta = \frac{1}{2}(a^2 + ab)\sin\theta,$$

which is constant.

Alternate Solution: Let O be the center of the circle and r its radius. Then $[CPQ] = [CDQPB] - [CDQ] - [CBP]$, where [...] denotes area of the polygon with given vertices. Note that $[CDQPB]$ is half r times the perimeter of $CDQPB$. Note that the heights of CDQ and CBP are $2r$ so $[CDQ] = r \cdot DQ$ and $[CBP] = r \cdot PB$. Using the fact that $QT = QV$ and $PU = PT$, it now follows that $[CPQ] = [OVDCBU] - [CDV] - [CBU]$, which is independent of P and Q.

□

5. A purse contains a finite number of coins, each with distinct positive integer values. Is it possible that there are exactly 2020 ways to use coins from the purse to make the value 2020?

Solution: It is possible.
Consider a coin purse with coins of values $2, 4, 8, 2014, 2016, 2018, 2020$ and every odd number between 503 and 1517. Call such a coin *big* if its value is between 503 and 1517. Call a coin *small* if its value is $2, 4$ or 8 and *huge* if its value is $2014, 2016, 2018$ or 2020. Suppose some subset of these coins contains no huge coins and sums to 2020. If it contains at least four big coins, then its value must be at least $503 + 505 + 507 + 509 > 2020$. Furthermore since all of the small coins are even in value, if the subset contains exactly one or three big coins, then its value must be odd. Thus the subset must contain exactly two big coins. The eight possible subsets of the small coins have values $0, 2, 4, 6, 8, 10, 12, 14$. Therefore the ways to make the value 2020 using no huge coins correspond to the pairs of big coins with sums $2006, 2008, 2010, 2012, 2014, 2016, 2018$ and 2020. The numbers of such pairs are $250, 251, 251, 252, 252, 253, 253, 254$, respectively. Thus there are exactly 2016 subsets of this coin purse with value 2020 using no huge coins. There are exactly four ways to make a value of 2020 using huge coins; these are $\{2020\}, \{2, 2018\}, \{4, 2016\}$ and $\{2, 4, 2014\}$. Thus there are exactly 2020 ways to make the value 2020.

Alternate construction: Take the coins $1, 2, \ldots, 11, 1954, 1955, \ldots, 2019$. The only way to get 2020 is a non-empty subset of $1, \ldots, 11$ and a single *large* coin. There are 2047 non-empty such subsets of sums between 1 and 66. Thus they each correspond to a unique large coin making 2020, so we have 2047 ways. Thus we only need to remove some large coins, so that we remove exactly 27 small sums. This can be done, for example, by removing coins $2020 - n$ for $n = 1, 5, 6, 7, 8, 9$, as these correspond to $1 + 3 + 4 + 5 + 6 + 8 = 27$ partitions into distinct numbers that are at most 11.

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