The 2019 Canadian Mathematical Olympiad

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

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Official Problem Set

1. Amy has drawn three points in a plane, A, B, and C, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter O of triangle ABC, and then afterwards she can draw the circumcenter of triangle ABO.

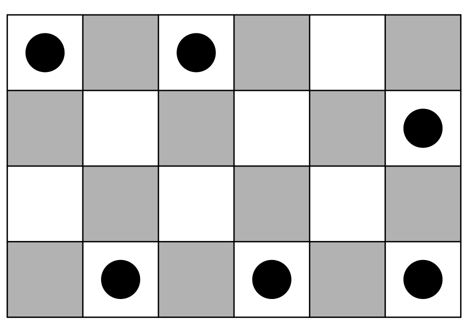
   (a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

   (b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

   (Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let a and b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

3. Let m and n be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing mn counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.



4. Let n be an integer greater than 1, and let $a_0, a_1, \ldots, a_n$ be real numbers with $a_1 = a_{n-1} = 0$. Prove that for any real number k,

$$|a_0| - |a_n| \le \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}| .$$

5. David and Jacob are playing a game of connecting $n \geq 3$ points drawn in a plane. No three of the points are collinear. On each player's turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the n given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all n for which he has a winning strategy.

Important!

Please do not discuss this problem set online for at least 24 hours.