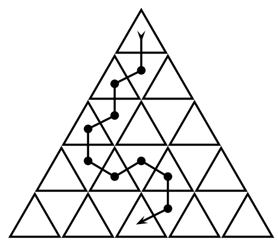
37th Canadian Mathematical Olympiad

Wednesday, March 30, 2005

1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

   a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

   b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

[Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]