THE 1999 CANADIAN MATHEMATICAL OLYMPIAD

1. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$.

   Here, if x is a real number, then $[x]$ denotes the greatest integer that is less than or equal to x.

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

3. Determine all positive integers n with the property that $n = (d(n))^2$. Here $d(n)$ denotes the number of positive divisors of n.

4. Suppose $a_1, a_2, \ldots, a_8$ are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find a specific set of 7 distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that the equation $a_i - a_j = k$ does not have three distinct solutions for any $k > 0$.

5. Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that

   $$x^2y + y^2z + z^2x \leq \frac{4}{27},$$

   and find when equality occurs.