THE 1998 CANADIAN MATHEMATICAL OLYMPIAD

1. Determine the number of real solutions a to the equation

$$\left[\frac{1}{2}\,a\right] + \left[\frac{1}{3}\,a\right] + \left[\frac{1}{5}\,a\right] = a\;.$$

Here, if x is a real number, then $[\,x\,]$ denotes the greatest integer that is less than or equal to x.

2. Find all real numbers x such that

$$x = \left(x - \frac{1}{x}\right)^{1/2} + \left(1 - \frac{1}{x}\right)^{1/2}.$$

3. Let n be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

4. Let ABC be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point of intersection of the lines BD and CE. Show that the line AF is perpendicular to the line BC.

5. Let m be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if (a, b) is of the form (a_n, a_{n+1}) for some $n \geq 0$.