Canadian Mathematical Olympiad
1997

PROBLEM 1

How many pairs of positive integers x, y are there, with $x \leq y$, and such that $\gcd(x, y) = 5!$ and $\operatorname{lcd}(x, y) = 50!$.

NOTE. $\gcd(x, y)$ denotes the greatest common divisor of x and y, $\operatorname{lcd}(x, y)$ denotes the least common multiple of x and y, and $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

PROBLEM 2

The closed interval $A = [0, 50]$ is the union of a finite number of closed intervals, each of length 1. Prove that some of the intervals can be removed so that those remaining are mutually disjoint and have total length ≥ 25.

NOTE. For $a \leq b$, the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ has length $b - a$; disjoint intervals have *empty* intersection.

PROBLEM 3

Prove that

$$\frac{1}{1999} < \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{1997}{1998} < \frac{1}{44}.$$

PROBLEM 4

The point O is situated inside the parallelogram $ABCD$ so that

$$\angle AOB + \angle COD = 180^\circ.$$

Prove that $\angle OBC = \angle ODC$.

PROBLEM 5

Write the sum

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{k^3 + 9k^2 + 26k + 24}$$

in the form $\frac{p(n)}{q(n)}$, where p and q are polynomials with integer coefficients.