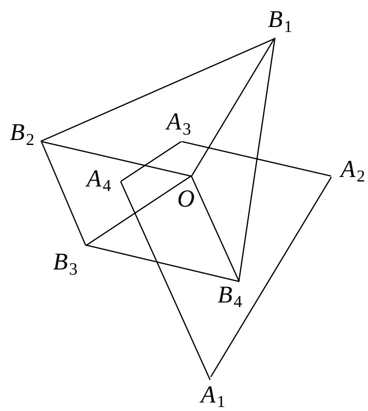
Canadian Mathematical Olympiad
1982

Problem 1

In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1, 2, 3$ and $4 (A_5 = A_1)$. Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



Problem 2

If a, b and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,

(i) show that a, b and c are distinct:

(ii) show that

$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer.

Problem 3

Let R^n be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at irrational distance from at least one point in that set.

Problem 4

Let p be a permutation of the set $S_n = \{1, 2, \ldots, n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

Problem 5

The altitudes of a tetrahedron $ABCD$ are extended externally to points A', B', C' and D' respectively, where $AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$ and $DD' =$

k/h_d. Here, k is a constant and h_a denotes the length of the altitude of $ABCD$ from vertex A, *etc.* Prove that the centroid of the tetrahedron $A'B'C'D'$ coincides with the centroid of $ABCD$.