Canadian Mathematical Olympiad
1979

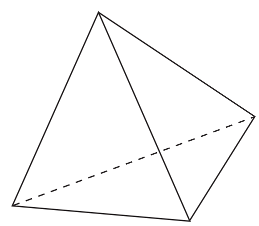
Problem 1

Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1 A_2 \geq G_1 G_2$$

Problem 2

It is known in Euclidean geometry that the sum of the angles of a triangle is constant. Prove, however, that the sum of the dihedral angles of a tetrahedron is *not* constant.



Note. (i) A tetrahedron is a triangular pyramid, and (ii) a dihedral angle is the interior angle between a pair of faces.

Problem 3

Let a, b, c, d, e be integers such that $1 \leq a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16},$$

where $[m, n]$ denotes the least common multiple of m and n (*e.g.* $[4, 6] = 12$).

Problem 4

A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

Problem 5

A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is *self-avoiding* if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$