SOLUTIONS FOR 2012 APMO PROBLEMS

Problem 1.

Solution: Let us denote by $\triangle XYZ$ the area of the triangle XYZ. Let $x = \triangle PAB$, $y = \triangle PBC$ and $z = \triangle PCA$.

From

$$y : z = \triangle BCP : \triangle ACP = BF : AF = \triangle BPF : \triangle APF = (x-1) : 1$$

follows that $z(x-1) = y$, which yields $(z+1)x = x+y+z$. Similarly, we get $(x+1)y = x+y+z$ and $(y+1)z = x+y+z$. Thus, we obtain $(x+1)y = (y+1)z = (z+1)x$.

We may assume without loss of generality that $x \leq y, z$. If we assume that $y > z$ holds, then we get $(y+1)z > (z+1)x$, which is a contradiction. Similarly, we see that $y < z$ leads to a contradiction $(x+1)y < (y+1)z$. Therefore, we must have $y = z$. Then, we also get from $(y+1)z = (z+1)x$ that $x = z$ must hold. We now obtain from $(x-1) : 1 = y : z = 1 : 1$ that $x = y = z = 2$ holds. Therefore, we conclude that the area of the triangle ABC equals $x+y+z = 6$.

Problem 2.

Solution: If we insert numbers as in the figure below ($0'$s are to be inserted in the remaining blank boxes), then we see that the condition of the problem is satisfied and the total number of all the numbers inserted is 5.

0	1	0	
1	1	1	
0	1	0	

We will show that the sum of all the numbers to be inserted in the boxes of the given grid cannot be more than 5 if the distribution of the numbers has to satisfy the requirement of the problem. Let $n = 2012$. Let us say that the row number (the column number) of a box in the given grid is i (j, respectively) if the box lies on the i-th row and the j-th column. For a pair of positive integers x and y, denote by $R(x,y)$ the sum of the numbers inserted in all of the boxes whose row number is greater than or equal to x and less than or equal to y (assign the value 0 if $x > y$).

First let a be the largest integer satisfying $1 \leq a \leq n$ and $R(1, a-1) \leq 1$, and then choose the smallest integer c satisfying $a \leq c \leq n$ and $R(c+1, n) \leq 1$. It is possible to choose such a pair a, c since $R(1,0) = 0$ and $R(n+1, n) = 0$. If $a < c$, then we have $a < n$ and so, by the maximality of a, we must have $R(1,a) > 1$, while from the minimality of c, we must have $R(a+1, n) > 1$. Then by splitting the grid into 2 rectangles by means of the horizontal line bordering the a-th row and the $a+1$-th row, we get the splitting contradicting the requirement of the problem. Thus, we must have $a = c$.

Similarly, if for any pair of integers x, y we define $C(x,y)$ to be the sum of the numbers inserted in all of the boxes whose column number is greater than or equal to x and less than or equal to y ($C(x,y) = 0$ if $x > y$), then we get a number b for which

$$C(1, b-1) \leq 1, \quad C(b+1, n) \leq 1, \quad 1 \leq b \leq n.$$

If we let r be the number inserted in the box whose row number is a and the column number is b, then since $r \leq 1$, we conclude that the sum of the numbers inserted into all of the boxes is

$$\leq R(1, a-1) + R(a+1, n) + C(1, b-1) + C(b+1, n) + r \leq 5.$$

Problem 3.

Solution

For integers a, b and a positive integer m, let us write $a \equiv b \pmod{m}$ if $a - b$ is divisible by m. Since $\frac{n^p+1}{p^n+1}$ must be a positive integer, we see that $p^n \leq n^p$ must hold. This means that if $p = 2$, then $2^n \leq n^2$ must hold. As it is easy to show by induction that $2^n > n^2$ holds if $n \geq 5$, we conclude that if $p = 2$, then $n \leq 4$ must be satisfied. And we can check that $(p, n) = (2, 2), (2, 4)$ satisfy the condition of the problem, while $(2, 3)$ does not.

Next, we consider the case where $p \geq 3$.

Suppose s is an integer satisfying $s \geq p$. If $s^p \leq p^s$ for such an s, then we have

$$\begin{aligned}
(s+1)^p = s^p\left(1+\frac{1}{s}\right)^p &\leq p^s\left(1+\frac{1}{p}\right)^p \\
&= p^s \sum_{r=0}^{p} {}_pC_r \frac{1}{p^r} < p^s \sum_{r=0}^{p} \frac{1}{r!} \\
&\leq p^s\left(1 + \sum_{r=1}^{p} \frac{1}{2^{r-1}}\right) \\
&< p^s(1+2) \leq p^{s+1}
\end{aligned}$$

Thus we have $(s+1)^p < p^{s+1}$, and by induction on n, we can conclude that if $n > p$, then $n^p < p^n$. This implies that we must have $n \leq p$ in order to satisfy our requirement $p^n \leq n^p$.

We note that since $p^n + 1$ is even, so is $n^p + 1$, which, in turn implies that n must be odd and therefore, $p^n + 1$ is divisible by $p + 1$, and $n^p + 1$ is also divisible by $p+1$. Thus we have $n^p \equiv -1 \pmod{(p+1)}$, and therefore, $n^{2p} \equiv 1 \pmod{(p+1)}$.

Now, let e be the smallest positive integer for which $n^e \equiv 1 \pmod{(p+1)}$. Then, we can write $2p = ex + y$, where x, y are non-negative integers and $0 \leq y < e$, and we have

$$1 \equiv n^{2p} = (n^e)^x \cdot n^y \equiv n^y \pmod{(p+1)},$$

which implies, because of the minimality of e, that $y = 0$ must hold. This means that $2p$ is an integral multiple of e, and therefore, e must equal one of the numbers $1, 2, p, 2p$.

Now, if $e = 1, p$, then we get $n^p \equiv 1 \pmod{(p+1)}$, which contradicts the fact that p is an odd prime. Since n and $p + 1$ are relatively prime, we have by Euler's Theorem that $n^{\varphi(p+1)} \equiv 1 \pmod{(p+1)}$, where $\varphi(m)$ denotes the number of integers $j (1 \leq j \leq m)$ which are relatively prime with m. From $\varphi(p+1) < p+1 < 2p$ and the minimality of e, we can then conclude that $e = 2$ must hold.

From $n^2 \equiv 1 \pmod{(p+1)}$, we get

$$-1 \equiv n^p = n^{(2 \cdot \frac{p-1}{2} + 1)} \equiv n \pmod{(p+1)},$$

which implies that $p+1$ divides $n+1$. Therefore, we must have $p \le n$, which, together with the fact $n \le p$, show that $p = n$ must hold.

It is clear that the pair (p,p) for any prime $p \ge 3$ satisfies the condition of the problem, and thus, we conclude that the pairs (p,n) which satisfy the condition of the problem must be of the form $(2,4)$ and (p,p) with any prime p.

Alternate Solution. Let us consider the case where $p \ge 3$. As we saw in the preceding solution, n must be odd if the pair (p,n) satisfy the condition of the problem. Now, let q be a prime factor of $p+1$. Then, since $p+1$ divides p^n+1, q must be a prime factor of p^n+1 and of n^p+1 as well. Suppose $q \ge 3$. Then, from $n^p \equiv -1 \pmod q$, it follows that $n^{2p} \equiv 1 \pmod q$ holds. If we let e be the smallest positive integer satisfying $n^e \equiv 1 \pmod q$, then by using the same argument as we used in the preceding solution, we can conclude that e must equal one of the numbers $1, 2, p, 2p$. If $e = 1, p$, then we get $n^p \equiv 1 \pmod q$, which contradicts the assumption $q \ge 3$. Since n is not a multiple of q, by Fermat's Little Theorem we get $n^{q-1} \equiv 1 \pmod q$, and therefore, we get by the minimality of e that $e = 2$ must hold. From $n^2 \equiv 1 \pmod q$, we also get

$$n^p = n^{(2\cdot\frac{p-1}{2}+1)} \equiv n \pmod q,$$

and since $n^p \equiv -1 \pmod q$, we have $n \equiv -1 \pmod q$ as well.

Now, if $q = 2$ then since n is odd, we have $n \equiv -1 \pmod q$ as well. Thus, we conclude that for an arbitrary prime factor q of $p+1$, $n \equiv -1 \pmod q$ must hold.

Suppose, for a prime q, q^k for some positive integer k is a factor of $p+1$. Then q^k must be a factor of n^p+1 as well. But since

$$n^p + 1 = (n+1)(n^{p-1} - n^{p-2} + \cdots - n + 1) \quad \text{and}$$

$$n^{p-1} - n^{p-2} + \cdots - n + 1 \equiv (-1)^{p-1} - (-1)^{p-2} + \cdots - (-1) + 1 \not\equiv 0 \pmod q,$$

we see that q^k must divide $n+1$. By applying the argument above for each prime factor q of $p+1$, we can then conclude that $n+1$ must be divisible by $p+1$, and as we did in the preceding proof, we can conclude that $n = p$ must hold.

Problem 4.

Solution: If $AB = AC$, then we get $BF = CF$ and the conclusion of the problem is clearly satisfied. So, we assume that $AB \ne AC$ in the sequel.

Due to symmetry, we may suppose without loss of generality that $AB > AC$. Let K be the point on the circle Γ such that AK is a diameter of this circle. Then, we get

$$\angle BCK = \angle ACK - \angle ACB = 90^\circ - \angle ACB = \angle CBH$$

and

$$\angle CBK = \angle ABK - \angle ABC = 90^\circ - \angle ABC = \angle BCH,$$

from which we conclude that the triangles BCK and CBH are congruent. Therefore, the quadrilateral $BKCH$ is a parallelogram, and its diagonal HK passes through the center M of the other diagonal BC. Therefore, the 3 points H, M, K lie on the same straight line, and we have $\angle AEM = \angle AEK = 90^\circ$.

From $\angle AED = 90^\circ = \angle ADM$, we see that the 4 points A, E, D, M lie on the circumference of the same circle, from which we obtain $\angle AMB = \angle AED = \angle AEF = \angle ACF$. Putting this fact together with the fact that $\angle ABM = \angle AFC$, we conclude that the triangles ABM and AFC are similar, and we get $\frac{AM}{BM} = \frac{AC}{FC}$. By a similar argument, we get that the triangles ACM and AFB are similar, and therefore, that $\frac{AM}{CM} = \frac{AB}{FB}$ holds. Noting that $BM = CM$, we also get $\frac{AC}{FC} = \frac{AB}{FB}$, from which we can conclude that $\frac{BF}{CF} = \frac{AB}{AC}$, proving the assertion of the problem.

Problem 5.

Solution: Let us note first that if $i \neq j$, then since $a_i a_j \leq \frac{a_i^2+a_j^2}{2}$, we have

$$n - a_i a_j \geq n - \frac{a_i^2 + a_j^2}{2} \geq n - \frac{n}{2} = \frac{n}{2} > 0.$$

If we set $b_i = |a_i|$ $(i = 1, 2, \dots, n)$, then we get $b_1^2 + b_2^2 + \cdots + b_n^2 = n$ and $\frac{1}{n-a_i a_j} \leq \frac{1}{n-b_i b_j}$, which shows that it is enough to prove the assertion of the problem in the case where all of $a_1, a_2, \cdots, a_n$ are non-negative. Hence, we assume from now on that $a_1, a_2, \cdots, a_n$ are all non-negative.

By multiplying by n the both sides of the desired inequality we get the inequality:

$$\sum_{1 \leq i < j \leq n} \frac{n}{n - a_i a_j} \leq \frac{n^2}{2}$$

and since $\frac{n}{n-a_i a_j} = 1 + \frac{a_i a_j}{n-a_i a_j}$, we obtain from the inequality above by subtracting $\frac{n(n-1)}{2}$ from both sides the following inequality:

$$\text{(i)} \qquad \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{n - a_i a_j} \leq \frac{n}{2}$$

We will show that this inequality (i) holds.

If for some i the equality $a_i^2 = n$ is valid, then $a_j = 0$ must hold for all $j \neq i$ and the inequality (i) is trivially satisfied. So, we assume from now on that $a_i^2 < n$ is valid for each i.

Let us assume that $i \neq j$ from now on. Since $0 \leq a_i a_j \leq \left(\frac{a_i+a_j}{2}\right)^2 \leq \frac{a_i^2+a_j^2}{2}$ holds, we have

$$\text{(ii)} \qquad \frac{a_i a_j}{n - a_i a_j} \leq \frac{a_i a_j}{n - \frac{a_i^2+a_j^2}{2}} \leq \frac{\left(\frac{a_i+a_j}{2}\right)^2}{n - \frac{a_i^2+a_j^2}{2}} = \frac{1}{2} \cdot \frac{(a_i + a_j)^2}{(n - a_i^2) + (n - a_j^2)}.$$

Since $n - a_i^2 > 0$, $n - a_j^2 > 0$, we also get from the Cauchy-Schwarz inequality that

$$\left(\frac{a_j^2}{n - a_i^2} + \frac{a_i^2}{n - a_j^2}\right)((n - a_i^2) + (n - a_j^2)) \geq (a_i + a_j)^2,$$

from which it follows that

$$\text{(iii)} \qquad \frac{(a_i + a_j)^2}{(n - a_i^2) + (n - a_j^2)} \leq \left(\frac{a_j^2}{n - a_i^2} + \frac{a_i^2}{n - a_j^2}\right)$$

holds. Combining the inequalities (ii) and (iii), we get

$$\begin{aligned}
\sum_{1\le i<j\le n} \frac{a_i a_j}{n - a_i a_j} &\le \frac{1}{2} \sum_{1\le i<j\le n} \left(\frac{a_j^2}{n - a_i^2} + \frac{a_i^2}{n - a_j^2} \right) \\
&= \frac{1}{2} \sum_{i\ne j} \frac{a_j^2}{n - a_i^2} \\
&= \frac{1}{2} \sum_{i=1}^{n} \frac{n - a_i^2}{n - a_i^2} \\
&= \frac{n}{2},
\end{aligned}$$

which establishes the desired inequality (i).