XXI Asian Pacific Mathematics Olympiad

APMO

March, 2009

Problem 1. Consider the following operation on positive real numbers written on a blackboard:

> Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

(Solution) Using AM-GM inequality, we obtain

$$\frac{1}{r^2} = \frac{2}{ab} = \frac{2ab}{a^2b^2} \leq \frac{a^2+b^2}{a^2b^2} \leq \frac{1}{a^2} + \frac{1}{b^2}\,. \tag{$*$}$$

Consequently, if we let S_ℓ be the sum of the squares of the reciprocals of the numbers written on the board after ℓ operations, then S_ℓ increases as ℓ increases, that is,

$$S_0 \leq S_1 \leq \cdots \leq S_{k^2-1}. \tag{$**$}$$

Therefore if we let s be the smallest real number written on the board after k^2-1 operations, then $\frac{1}{s^2} \geq \frac{1}{t^2}$ for any number t among k^2 numbers on the board and hence

$$k^2 \times \frac{1}{s^2} \geq S_{k^2-1} \geq S_0 = \frac{1}{r^2}\,,$$

which implies that $s \leq kr$ as desired. $\square$

Remark. The nature of the problem does not change at all if the numbers on the board are restricted to be positive integers. But that may mislead some contestants to think the problem is a number theoretic problem rather than a combinatorial problem.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2} \quad \text{for } k=1,2,3,4,5.$$

Find the value of $\dfrac{a_1}{37}+\dfrac{a_2}{38}+\dfrac{a_3}{39}+\dfrac{a_4}{40}+\dfrac{a_5}{41}$. (Express the value in a single fraction.)

(Solution) Let $R(x) := \dfrac{a_1}{x^2+1}+\dfrac{a_2}{x^2+2}+\dfrac{a_3}{x^2+3}+\dfrac{a_4}{x^2+4}+\dfrac{a_5}{x^2+5}$. Then $R(\pm 1)=1$, $R(\pm 2)=\dfrac{1}{4}$, $R(\pm 3)=\dfrac{1}{9}$, $R(\pm 4)=\dfrac{1}{16}$, $R(\pm 5)=\dfrac{1}{25}$ and $R(6)$ is the value to be found. Let's put $P(x) := (x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)$ and $Q(x) := R(x)P(x)$. Then for $k=\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get $Q(k)=R(k)P(k)=\dfrac{P(k)}{k^2}$, that is, $P(k)-k^2Q(k)=0$. Since $P(x)-x^2Q(x)$ is a polynomial of degree 10 with roots $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get

$$P(x)-x^2Q(x)=A(x^2-1)(x^2-4)(x^2-9)(x^2-16)(x^2-25). \qquad (*)$$

Putting $x=0$, we get $A=\dfrac{P(0)}{(-1)(-4)(-9)(-16)(-25)}=-\dfrac{1}{120}$. Finally, dividing both sides of $(*)$ by $P(x)$ yields

$$1-x^2R(x)=1-x^2\frac{Q(x)}{P(x)}=-\frac{1}{120}\cdot\frac{(x^2-1)(x^2-4)(x^2-9)(x^2-16)(x^2-25)}{(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)}$$

and hence that

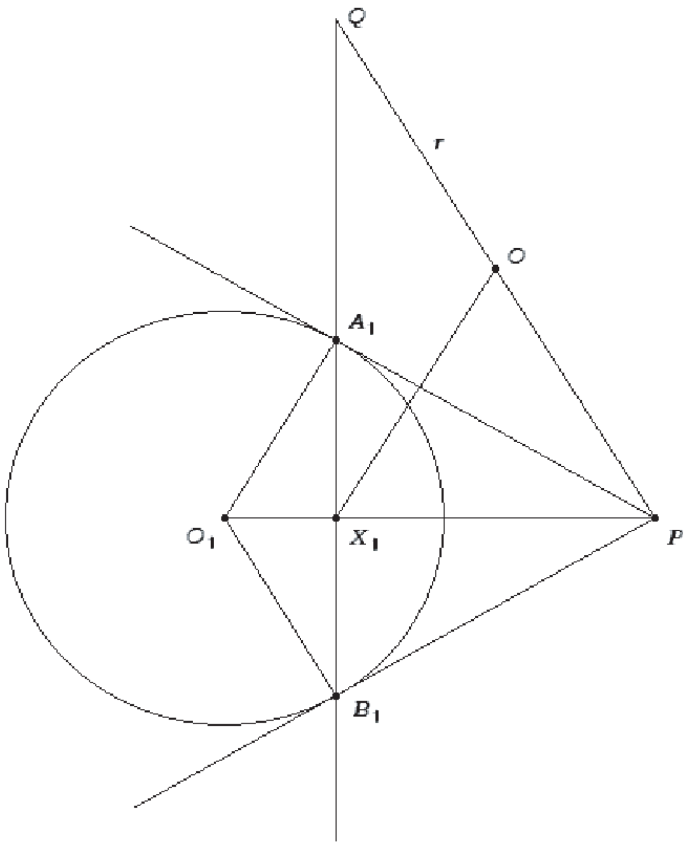
$$1-36R(6)=-\frac{35\times 32\times 27\times 20\times 11}{120\times 37\times 38\times 39\times 40\times 41}=-\frac{3\times 7\times 11}{13\times 19\times 37\times 41}=-\frac{231}{374699},$$

which implies $R(6)=\dfrac{187465}{6744582}$.

Remark. We can get $a_1=\dfrac{1105}{72}, a_2=-\dfrac{2673}{40}, a_3=\dfrac{1862}{15}, a_4=-\dfrac{1885}{18}, a_5=\dfrac{1323}{40}$ by solving the given system of linear equations, which is extremely messy and takes a lot of time.

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

(Solution) Let O_i be the center and r_i the radius of circle Γ_i for each $i = 1, 2, 3$. Let P be an exceptional point, and let the three corresponding lines A_1B_1, A_2B_2,A_3B_3 concur at Q. Construct the circle with diameter PQ. Call the circle Γ, its center O and its radius r. We now claim that all exceptional points lie on Γ.



Let PO_1 intersect A_1B_1 in X_1. As $PO_1 \perp A_1B_1$, we see that X_1 lies on Γ. As PA_1 is a tangent to Γ_1, triangle PA_1O_1 is right-angled and similar to triangle $A_1X_1O_1$. It follows that

$$\frac{O_1X_1}{O_1A_1} = \frac{O_1A_1}{O_1P}, \quad \text{i.e.,} \quad O_1X_1 \cdot O_1P = O_1A_1{}^2 = r_1{}^2.$$

On the other hand, $O_1X_1 \cdot O_1P$ is also the power of O_1 with respect to Γ, so that

$$r_1^2 = O_1X_1 \cdot O_1P = (O_1O - r)(O_1O + r) = O_1O^2 - r^2, \tag{$*$}$$

and hence

$$r^2 = OO_1^2 - r_1^2 = (OO_1 - r_1)(OO_1 + r_1).$$

Thus, r^2 is the power of O with respect to Γ_1. By the same token, r^2 is also the power of O with respect to Γ_2 and Γ_3. Hence O must be the radical center of the three given circles. Since r, as the square root of the power of O with respect to the three given circles, does not depend on P, it follows that all exceptional points lie on Γ. $\square$

Remark. In the event of the radical point being at infinity (and hence the three radical axes being parallel), there are no exceptional points in the plane, which is consistent with the statement of the problem.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

(Solution) For $k = 1$, there is nothing to prove. Henceforth assume $k \geq 2$.
Let $p_1, p_2, \ldots, p_k$ be k distinct primes such that

$$k < p_k < \cdots < p_2 < p_1$$

and let $N = p_1 p_2 \cdots p_k$. By Chinese Remainder Theorem, there exists a positive integer x satisfying

$$x \equiv -i \pmod{p_i}$$

for all $i = 1, 2, \ldots, k$ and $x > N^2$. Consider the following sequence:

$$\frac{x+1}{N}, \quad \frac{x+2}{N}, \quad , \ldots, \quad \frac{x+k}{N}.$$

This sequence is obviously an arithmetic sequence of positive rational numbers of length k. For each $i = 1, 2, \ldots, k$, the numerator $x + i$ is divisible by p_i but not by p_j for $j \neq i$, for otherwise p_j divides $|i - j|$, which is not possible because $p_j > k > |i - j|$. Let

$$a_i := \frac{x+i}{p_i}, \quad b_i := \frac{N}{p_i} \qquad \text{for all } \; i = 1, 2, \ldots, k.$$

Then

$$\frac{x+i}{N} = \frac{a_i}{b_i}, \quad \gcd(a_i, b_i) = 1 \qquad \text{for all } \; i = 1, 2, \ldots, k,$$

and all b_i's are distinct from each other. Moreover, $x > N^2$ implies

$$a_i = \frac{x+i}{p_i} > \frac{N^2}{p_i} > N > \frac{N}{p_j} = b_j \qquad \text{for all } \; i, j = 1, 2, \ldots, k$$

and hence all a_i's are distinct from b_i's. It only remains to show that all a_i's are distinct from each other. This follows from

$$a_j = \frac{x+j}{p_j} > \frac{x+i}{p_j} > \frac{x+i}{p_i} = a_i \qquad \text{for all } \; i < j$$

by our choice of $p_1, p_2, \ldots, p_k$. Thus, the arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of positive rational numbers satisfies the conditions of the problem. $\square$

Remark. Here is a much easier solution :

For any positive integer $k \geq 2$, consider the sequence

$$\frac{(k!)^2+1}{k!}, \frac{(k!)^2+2}{k!}, \ldots, \frac{(k!)^2+k}{k!}.$$

Note that $\gcd(k!, (k!)^2+i) = i$ for all $i = 1, 2, \ldots, k$. So, taking

$$a_i := \frac{(k!)^2+i}{i}, \quad b_i := \frac{k!}{i} \qquad \text{for all} \quad i = 1, 2, \ldots, k,$$

we have $\gcd(a_i, b_i) = 1$ and

$$a_i = \frac{(k!)^2+i}{i} > a_j = \frac{(k!)^2+j}{j} > b_i = \frac{k!}{i} > b_j = \frac{k!}{j}$$

for any $1 \leq i < j \leq k$. Therefore this sequence satisfies every condition given in the problem.

Problem 5. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a $90°$ left turn after every ℓ kilometer driving from start; Rob makes a $90°$ right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

(Solution) Let Zillis be d kilometers away from Argovia, where d is a positive real number. For simplicity, we will position Argovia at $(0, 0)$ and Zillis at $(d, 0)$, so that the car starts out facing east. We will investigate how the car moves around in the period of travelling the first ℓr kilometers, the second ℓr kilometers, $\ldots$, and so on. We call each period of travelling ℓr kilometers a *section*. It is clear that the car will have identical behavior in every section except the direction of the car at the beginning.

`Case 1:` $\underline{\ell - r \equiv 2 \pmod 4}$. After the first section, the car has made $\ell - 1$ right turns and $r - 1$ left turns, which is a net of $2 (\equiv \ell - r \pmod 4)$ right turns. Let the displacement vector for the first section be (x, y). Since the car has rotated $180°$, the displacement vector for the second section will be $(-x, -y)$, which will take the car back to $(0, 0)$ facing east again. We now have our original situation, and the car has certainly never travelled further than ℓr kilometers from Argovia. So, the car cannot reach Zillis if it is further apart from Argovia.

`Case 2:` $\underline{\ell - r \equiv 1 \pmod 4}$. After the first section, the car has made a net of 1 right turn. Let the displacement vector for the first section again be (x, y). This time the car has rotated $90°$ clockwise. We can see that the displacements for the second, third and fourth section will be $(y, -x)$, $(-x, -y)$ and $(-y, x)$, respectively, so after four sections the car is back at $(0, 0)$ facing east. Since the car has certainly never travelled further than $2\ell r$ kilometers from Argovia, the car cannot reach Zillis if it is further apart from Argovia.

`Case 3:` $\underline{\ell - r \equiv 3 \pmod 4}$. An argument similar to that in `Case 2` (switching the roles of left and right) shows that the car cannot reach Zillis if it is further apart from Argovia.

`Case 4:` $\underline{\ell \equiv r \pmod 4}$. The car makes a net turn of $0°$ after each section, so it must be facing east. We are going to show that, after traversing the first section, the car will be at $(1, 0)$. It will be useful to interpret the Cartesian plane as the complex plane, i.e. writing $x + iy$ for (x, y), where $i = \sqrt{-1}$. We will denote the k-th kilometer of movement by m_{k-1}, which takes values from the set $\{1, i, -1, -i\}$, depending on the direction. We then just have to show that

$$\sum_{k=0}^{\ell r-1} m_k = 1,$$

which implies that the car will get to Zillis no matter how far it is apart from Argovia.

`Case 4a:` $\underline{\ell \equiv r \equiv 1 \pmod 4}$. First note that for $k = 0, 1, \ldots, \ell r - 1$,

$$m_k = i^{\lfloor k/\ell \rfloor}(-i)^{\lfloor k/r \rfloor}$$

since $\lfloor k/\ell \rfloor$ and $\lfloor k/r \rfloor$ are the exact numbers of left and right turns before the $(k+1)$st kilometer, respectively. Let $a_k (\equiv k \pmod \ell)$ and $b_k (\equiv k \pmod r)$ be the remainders of k when divided by ℓ and r, respectively. Then, since

$$a_k = k - \left\lfloor \frac{k}{\ell} \right\rfloor \ell \equiv k - \left\lfloor \frac{k}{\ell} \right\rfloor \pmod 4 \quad \text{and} \quad b_k = k - \left\lfloor \frac{k}{r} \right\rfloor r \equiv k - \left\lfloor \frac{k}{r} \right\rfloor \pmod 4,$$

we have $\lfloor k/\ell \rfloor \equiv k - a_k \pmod 4$ and $\lfloor k/r \rfloor \equiv k - b_k \pmod 4$. We therefore have

$$m_k = i^{k-a_k}(-i)^{k-b_k} = (-i^2)^k i^{-a_k}(-i)^{-b_k} = (-i)^{a_k}\, i^{b_k}\,.$$

As ℓ and r are relatively prime, by Chinese Remainder Theorem, there is a bijection between pairs $(a_k, b_k) = (k (\bmod \ell), k (\bmod r))$ and the numbers $k = 0, 1, 2, \ldots, \ell r - 1$. Hence

$$\sum_{k=0}^{\ell r-1} m_k = \sum_{k=0}^{\ell r-1} (-i)^{a_k}\, i^{b_k} = \left(\sum_{k=0}^{\ell-1} (-i)^{a_k}\right)\left(\sum_{k=0}^{r-1} i^{b_k}\right) = 1 \times 1 = 1$$

as required because $\ell \equiv r \equiv 1 \pmod 4$.

`Case 4b:` $\underline{\ell \equiv r \equiv 3 \pmod 4}$. In this case, we get

$$m_k = i^{a_k}(-i)^{b_k},$$

where $a_k (\equiv k \pmod \ell)$ and $b_k (\equiv k \pmod r)$ for $k = 0, 1, \ldots, \ell r - 1$. Then we can proceed analogously to `Case 4a` to obtain

$$\sum_{k=0}^{\ell r-1} m_k = \sum_{k=0}^{\ell r-1} (-i)^{a_k}\, i^{b_k} = \left(\sum_{k=0}^{\ell-1} (-i)^{a_k}\right)\left(\sum_{k=0}^{r-1} i^{b_k}\right) = i \times (-i) = 1$$

as required because $\ell \equiv r \equiv 3 \pmod 4$.

Now clearly the car traverses through all points between $(0,0)$ and $(1,0)$ during the first section and, in fact, covers all points between $(n-1, 0)$ and $(n, 0)$ during the n-th section. Hence it will eventually reach $(d, 0)$ for any positive d.

To summarize: (ℓ, r) satisfies the required conditions if and only if

$$\ell \equiv r \equiv 1 \quad \text{or} \quad \ell \equiv r \equiv 3 \pmod 4. \qquad \square$$

Remark. In case $\gcd(\ell, r) = d \neq 1$, the answer is:

$$\frac{\ell}{d} \equiv \frac{r}{d} \equiv 1 \quad \text{or} \quad \frac{\ell}{d} \equiv \frac{r}{d} \equiv 3 \pmod 4.$$