Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $\mid a_i - \frac{1}{2} \mid \geq f(n)$.

(Solution) The answer is

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

First, assume that n is even. If $a_i = \frac{1}{2}$ for all i, then the sum $a_1 + a_2 + \cdots + a_n$ is an integer. Since $\mid a_i - \frac{1}{2} \mid = 0$ for all i, we may conclude $f(n) = 0$ for any even n.

Now assume that n is odd. Suppose that $\mid a_i - \frac{1}{2} \mid < \frac{1}{2n}$ for all $1 \leq i \leq n$. Then, since $\sum_{i=1}^{n} a_i$ is an integer,

$$\frac{1}{2} \leq \left| \sum_{i=1}^{n} a_i - \frac{n}{2} \right| \leq \sum_{i=1}^{n} \left| a_i - \frac{1}{2} \right| < \frac{1}{2n} \cdot n = \frac{1}{2},$$

a contradiction. Thus $|a_i - \frac{1}{2}| \geq \frac{1}{2n}$ for some i, as required. On the other hand, putting $n = 2m + 1$ and $a_i = \frac{m}{2m+1}$ for all i gives $\sum a_i = m$, while

$$\left| a_i - \frac{1}{2} \right| = \frac{1}{2} - \frac{m}{2m+1} = \frac{1}{2(2m+1)} = \frac{1}{2n}$$

for all i. Therefore, $f(n) = \frac{1}{2n}$ is the best possible for any odd n.

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i, where i is an integer (not necessarily positive).

(Solution) We will prove this statement by induction using the equality

$$\tau^2 = \tau + 1.$$

If $n = 1$, then $1 = \tau^0$. Suppose that $n-1$ can be written as a finite sum of integral powers of τ, say

$$n - 1 = \sum_{i=-k}^{k} a_i \tau^i, \tag{1}$$

where $a_i \in \{0, 1\}$ and $n \geq 2$. We will write (1) as

$$n - 1 = a_k \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-k}. \tag{2}$$

For example,

$$1 = 1.0 = 0.11 = 0.1011 = 0.101011.$$

Firstly, we will prove that we may assume that in (2) we have $a_i a_{i+1} = 0$ for all i with $-k \leq i \leq k-1$. Indeed, if we have several occurrences of 11, then we take the leftmost such occurrence. Since we may assume that it is preceded by a 0, we can replace 011 with 100 using the identity $\tau^{i+1} + \tau^{i} = \tau^{i+2}$. By doing so repeatedly, if necessary, we will eliminate all occurrences of two 1's standing together. Now we have the representation

$$n - 1 = \sum_{i=-K}^{K} b_i \tau^{i}, \tag{3}$$

where $b_i \in \{0, 1\}$ and $b_i b_{i+1} = 0$.

If $b_0 = 0$ in (3), then we just add $1 = \tau^0$ to both sides of (3) and we are done.

Suppose now that there is 1 in the unit position of (3), that is $b_0 = 1$. If there are two 0's to the right of it, i.e.

$$n - 1 = \cdots 1.00 \cdots ,$$

then we can replace 1.00 with 0.11 because $1 = \tau^{-1} + \tau^{-2}$, and we are done because we obtain 0 in the unit position. Thus we may assume that

$$n - 1 = \cdots 1.010 \cdots .$$

Again, if we have $n - 1 = \cdots 1.0100 \cdots$, we may rewrite it as

$$n - 1 = \cdots 1.0100 \cdots = \cdots 1.0011 \cdots = \cdots 0.1111 \cdots$$

and obtain 0 in the unit position. Therefore, we may assume that

$$n - 1 = \cdots 1.01010 \cdots .$$

Since the number of 1's is finite, eventually we will obtain an occurrence of 100 at the end, i.e.

$$n - 1 = \cdots 1.01010 \cdots 100.$$

Then we can shift all 1's to the right to obtain 0 in the unit position, i.e.

$$n - 1 = \cdots 0.11 \cdots 11,$$

and we are done.

Problem 3. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

(Solution) Note that $r = \binom{p^2}{p} - p$. Hence, it suffices to show that

$$(p^2 - 1)(p^2 - 2) \cdots (p^2 - (p-1)) - (p-1)! \equiv 0 \pmod{p^4}. \tag{1}$$

Now, let

$$f(x) := (x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} + s_{p-2}x^{p-2} + \cdots + s_1 x + s_0. \qquad (2)$$

Then the congruence equation (1) is same as $f(p^2) - s_0 \equiv 0 \pmod{p^4}$. Therefore, it suffices to show that $s_1 p^2 \equiv 0 \pmod{p^4}$ or $s_1 \equiv 0 \pmod{p^2}$.

Since $a^{p-1} \equiv 1 \pmod{p}$ for all $1 \le a \le p-1$, we can factor

$$x^{p-1} - 1 \equiv (x-1)(x-2)\cdots(x-(p-1)) \pmod{p}. \qquad (3)$$

Comparing the coefficients of the left hand side of (3) with those of the right hand side of (2), we obtain $p \,|\, s_i$ for all $1 \le i \le p-2$ and $s_0 \equiv -1 \pmod{p}$. On the other hand, plugging p for x in (2), we get

$$f(p) = (p-1)! = s_0 = p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_1 p + s_0,$$

which implies

$$p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_2 p^2 = -s_1 p.$$

Since $p \ge 5$, $p \,|\, s_2$ and hence $s_1 \equiv 0 \pmod{p^2}$ as desired.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

(Solution) Let S be the tangent point of the circles O and O_1 and let T be the intersection point, different from S, of the circle O and the line SP. Let X be the tangent point of ℓ to O_1 and let M be the midpoint of the line segment XP. Since $\angle TBP = \angle ASP$, the triangle TBP is similar to the triangle ASP. Therefore,

$$\frac{PT}{PB} = \frac{PA}{PS}.$$

Since the line ℓ is tangent to the circle O_1 at X, we have

$$\angle SPX = 90^\circ - \angle XSP = 90^\circ - \angle APM = \angle PAM$$

which implies that the triangle PAM is similar to the triangle SPX. Consequently,

$$\frac{XS}{XP} = \frac{MP}{MA} = \frac{XP}{2MA} \quad \text{and} \quad \frac{XP}{PS} = \frac{MA}{AP}.$$

From this and the above observation follows

$$\frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{XP}{2MA} \cdot \frac{PA}{PS} = \frac{XP}{2MA} \cdot \frac{MA}{XP} = \frac{1}{2}. \tag{1}$$

Let A' be the intersection point of the circle O and the perpendicular bisector of the chord BC such that A, A' are on the same side of the line BC, and N be the intersection point of the lines $A'Q$ and CT. Since

$$\angle NCQ = \angle TCB = \angle TCA = \angle TBA = \angle TBP$$

and

$$\angle CA'Q = \frac{\angle CAB}{2} = \frac{\angle XAP}{2} = \angle PAM = \angle SPX,$$

the triangle NCQ is similar to the triangle TBP and the triangle $CA'Q$ is similar to the triangle SPX. Therefore

$$\frac{QN}{QC} = \frac{PT}{PB} \quad \text{and} \quad \frac{QC}{QA'} = \frac{XS}{XP}.$$

and hence $QA' = 2QN$ by (1). This implies that N is the midpoint of the line segment QA'. Let the circle O_2 touch the line segment AC at Y. Since

$$\angle ACN = \angle ACT = \angle BCT = \angle QCN$$

and $|CY| = |CQ|$, the triangles YCN and QCN are congruent and hence $NY \perp AC$ and $NY = NQ = NA'$. Therefore, N is the center of the circle O_2, which completes the proof.

Remark: Analytic solutions are possible: For example, one can prove for a triangle ABC inscribed in a circle O that $AB = k(2+2t), AC = k(1+2t), BC = k(1+4t)$ for some positive numbers k, t if and only if there exists a circle O_1 such that O_1 is tangent to the side AB at its midpoint, the side AC and the circle O.

One obtains $AB = k'(1+4t'), AC = k'(1+2t'), BC = k'(2+2t')$ by substituting $t = 1/4t'$ and $k = 2k't'$. So, there exists a circle O_2 such that O_2 is tangent to the side BC at its midpoint, the side AC and the circle O.

In the above, $t = \tan^2\alpha$ and $k = \frac{4R\tan\alpha}{(1+\tan^2\alpha)(1+4\tan^2\alpha)}$, where R is the radius of O and $\angle A = 2\alpha$. Furthermore, $t' = \tan^2\gamma$ and $k' = \frac{4R\tan\gamma}{(1+\tan^2\gamma)(1+4\tan^2\gamma)}$, where $\angle C = 2\gamma$. Observe that $\sqrt{tt'} = \tan\alpha \cdot \tan\gamma = \frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{1}{2}$, which implies $tt' = \frac{1}{4}$. It is now routine easy to check that $k = 2k't'$.

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

(Solution) Let C be the set of n clowns. Label the colours $1, 2, 3, \ldots, 12$. For each $i = 1, 2, \ldots, 12$, let E_i denote the set of clowns who use colour i. For each subset S of $\{1, 2, \ldots, 12\}$, let E_S be the set of clowns who use exactly those colours in S. Since $S \neq S'$ implies $E_S \cap E_{S'} = \emptyset$, we have

$$\sum_S |E_S| = |C| = n,$$

where S runs over all subsets of $\{1, 2, \ldots, 12\}$. Now for each i,

$$E_S \subseteq E_i \quad \text{if and only if} \quad i \in S,$$

and hence

$$|E_i| = \sum_{i \in S} |E_S|.$$

By assumption, we know that $|E_i| \leq 20$ and that if $E_S \neq \emptyset$, then $|S| \geq 5$. From this we obtain

$$20 \times 12 \geq \sum_{i=1}^{12} |E_i| = \sum_{i=1}^{12} \left(\sum_{i \in S} |E_S| \right) \geq 5 \sum_S |E_S| = 5n.$$

Therefore $n \leq 48$.

Now, define a sequence $\{c_i\}_{i=1}^{52}$ of colours in the following way:

$$\begin{array}{l} 1\ 2\ 3\ 4 \mid 5\ 6\ 7\ 8 \mid 9\ 10\ 11\ 12 \mid \\ 4\ 1\ 2\ 3 \mid 8\ 5\ 6\ 7 \mid 12\ 9\ 10\ 11 \mid \\ 3\ 4\ 1\ 2 \mid 7\ 8\ 5\ 6 \mid 11\ 12\ 9\ 10 \mid \\ 2\ 3\ 4\ 1 \mid 6\ 7\ 8\ 5 \mid 10\ 11\ 12\ 9 \mid 1\ 2\ 3\ 4 \end{array}$$

The first row lists $c_1, \ldots, c_{12}$ in order, the second row lists $c_{13}, \ldots, c_{24}$ in order, the third row lists $c_{25}, \ldots, c_{36}$ in order, and finally the last row lists $c_{37}, \ldots, c_{52}$ in order. For each j, $1 \leq j \leq 48$, assign colours $c_j, c_{j+1}, c_{j+2}, c_{j+3}, c_{j+4}$ to the j-th clown. It is easy to check that this assignment satisfies all conditions given above. So, 48 is the largest for n.

Remark: The fact that $n \leq 48$ can be obtained in a much simpler observation that

$$5n \leq 12 \times 20 = 240.$$

There are many other ways of constructing 48 distinct sets consisting of 5 colours. For example, consider the sets

$$\begin{array}{cccc} \{1,2,3,4,5,6\}, & \{3,4,5,6,7,8\}, & \{5,6,7,8,9,10\}, & \{7,8,9,10,11,12\}, \\ \{9,10,11,12,1,2\}, & \{11,12,1,2,3,4\}, & \{1,2,5,6,9,10\}, & \{3,4,7,8,11,12\}. \end{array}$$

Each of the above 8 sets has 6 distinct subsets consisting of exactly 5 colours. It is easy to check that the 48 subsets obtained in this manner are all distinct.