XXXI Asian Pacific Mathematics Olympiad

March, 2019

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo.ommenlinea.org`.
Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.

Problem 1. Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a and b.

Problem 2. Let m be a fixed positive integer. The infinite sequence $\{a_n\}_{n\geq 1}$ is defined in the following way: a_1 is a positive integer, and for every integer $n \geq 1$ we have

$$a_{n+1} = \begin{cases} a_n^2 + 2^m & \text{if } a_n < 2^m \\ a_n/2 & \text{if } a_n \geq 2^m. \end{cases}$$

For each m, determine all possible values of a_1 such that every term in the sequence is an integer.

Problem 3. Let ABC be a scalene triangle with circumcircle Γ. Let M be the midpoint of BC. A variable point P is selected in the line segment AM. The circumcircles of triangles BPM and CPM intersect Γ again at points D and E, respectively. The lines DP and EP intersect (a second time) the circumcircles to triangles CPM and BPM at X and Y, respectively. Prove that as P varies, the circumcircle of $\triangle AXY$ passes through a fixed point T distinct from A.

Problem 4. Consider a 2018×2019 board with integers in each unit square. Two unit squares are said to be neighbours if they share a common edge. In each turn, you choose some unit squares. Then for each chosen unit square the average of all its neighbours is calculated. Finally, after these calculations are done, the number in each chosen unit square is replaced by the corresponding average. Is it always possible to make the numbers in all squares become the same after finitely many turns?

Problem 5. Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + f(y)) = f(f(x)) + f(y^2) + 2f(xy)$$

for all real numbers x and y.