XXII Asian Pacific Mathematics Olympiad

March, 2010

Time allowed: 4 hours — *Each problem is worth 7 points*

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

Problem 4. Let ABC be an acute triangle satisfying the condition $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and the orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 5. Find all functions f from the set $\mathbf{R}$ of real numbers into $\mathbf{R}$ which satisfy for all $x, y, z \in \mathbf{R}$ the identity

$$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz).$$