XIX Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

∗ *The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that

$$2\angle AHI = 3\angle ABC.$$

Problem 3. Consider n disks $C_1, C_2, \ldots, C_n$ in a plane such that for each $1 \le i < n$, the center of C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \ge 1.$$

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.