XVII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

∗ *The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}\,.$$

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i-k| + |j-\ell| = 1$.

Problem 5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.