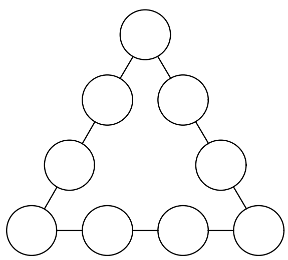
12th Asian Pacific Mathematics Olympiad

March 2000

Time allowed: 4 hours.
No calculators to be used.
Each question is worth 7 points.

1. Compute the sum $S=\sum_{i=0}^{101}\frac{x_i^3}{1-3x_i+3x_i^2}$ for $x_i=\frac{i}{101}$.

2. Given the following triangular arrangement of circles:



Each of the numbers 1, 2, …, 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

3. Let *ABC* be a triangle. Let *M* and *N* be the points in which the median and the angle bisector, respectively, at *A* meet the side *BC*. Let *Q* and *P* be the points in which the perpendicular at *N* to *NA* meets *MA* and *BA*, respectively, and *O* the point in which the perpendicular at *P* to *BA* meets *AN* produced. Prove that *QO* is perpendicular to *BC*.

4. Let *n*, *k* be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1}\cdot\frac{n^n}{k^k(n-k)^{n-k}}<\frac{n!}{k!(n-k)!}<\frac{n^n}{k^k(n-k)^{n-k}}.$$

5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence 0, 1, …, *n*. A transposition of a_i with a_j is called *legal* if $a_i=0$ for $i>0$, and $a_{i-1}+1=a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers *n* is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

END OF PAPER