Solutions for January

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that
$$\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$$
and
$$\frac{AE}{EC} \leq \frac{AF}{FB} \quad .$$
Prove that $[DEF] \leq \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

Solution 1. Let $BF = \mu BA$, $BD = \lambda BC$ and $CE = \nu CA$.

The conditions are that
$$\lambda \leq \mu \leq \frac{1}{2} \qquad \text{and} \qquad 1 - \nu \leq 1 - \mu \quad \text{or} \quad \mu \leq \nu \ .$$

We observe that $[BDF] = \lambda\mu[ABC]$.

To see this, let $BG = \lambda BA$. Then
$$[BDF] = \frac{\mu}{\lambda}[BGD] = \frac{\mu}{\lambda}\lambda^2[ABC] = \mu\lambda[ABC] \ .$$

Similarly $[AFE] = (1-\mu)(1-\lambda)[ABC]$ and $[DEC] = \nu(1-\lambda)[ABC]$.

Hence
$$\begin{aligned}
[DEF] &= (1 - \lambda\mu - (1-\mu)(1-\nu) - \nu(1-\lambda))[ABC] \\
&= (\mu - \mu\nu - \mu\lambda + \nu\lambda)[ABC] \\
&= \left(\frac{1}{4} - (\frac{1}{2} - \mu)^2 - (\mu - \lambda)(\nu - \mu)\right)[ABC] \leq \frac{1}{4}[ABC]
\end{aligned}$$
with equality if and only if $\mu = 1/2$ and either $\lambda = \mu = 1/2$ or $\nu = \mu = 1/2$. The result follows.

Solution 2. Let G be on AC so that $FG \| BC$. Then, since $\frac{AE}{EC} \leq \frac{AF}{FB}$, E lies in the segment AG.

Since $\frac{BD}{DC} \leq \frac{BF}{FA}$, DF produced is either parallel to AC or meets CA produced at a point X beyond A. Hence the distance from G to FD is not less than the distance from E to FD, so that $[DEF] \leq [FGD]$. The area of $[FGD]$ does not change as D varies along BC. To maximize $[DEF]$ is suffices to consider the special case of triangle $[FGD]$. Let $AF = xAB$. Then $FG = xBC$ and the heights of ΔDFG and ΔABC are in the ratio $1 - x$. Hence
$$\frac{[DFG]}{[ABC]} = x(1-x)$$
which is maximized when $x = \frac{1}{2}$. The result follows from this, with $[DEF]$ being exactly one quarter of $[ABC]$ when F and G are the midpoints of AB and AC respectively.

Solution 3. Set up the situation as in the second solution. Let $BF = tFA$. Then $AB = (1+t)FA$, and the height of the triangle FGD is $t/(1+t)$ times the height of the triangle ABC. Hence
$$[DEF] \leq [FGD] = \frac{t}{(1+t)^2}[ABC] \ .$$

Now
$$\frac{1}{4} - \frac{t}{(1+t)^2} = \frac{(1-t)^2}{4(1+t)^2} \geq 0$$

so that $t(1+t)^{-2} \leq 1/4$ and the result follows. Equality occurs if and only if $t = 1$ and $E = G$, *i.e.*, F and E are both midpoints of their sides.

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

(a) *Solution.* Recall that the centroid lies two-thirds of the way along the median from a vertex of the triangle to its opposite side. Let ABC be the fixed triangle and let $PQ \| BC$, $RS \| AC$ and $TU \| BA$ with PQ, RS and TU intersecting in the centroid G.

Observe, for example, that if A, X, Y are collinear and X and Y lie on PQ and BC respectively, then $AX : XY = 2 : 3$. It follows from this that, if one ant is at A, then the centroid of the triangle formed by the three ants lies inside ΔAPQ (otherwise the midpoint of the side opposite the ant at A would not be in ΔABC). Similarly, if one ant is at B (respectively C) then the centroid of the ants' triangle lies within ΔBRS (respectively ΔCTU). Thus, if one ant traverses the entire perimeter, the centroid of the ants' triangle must lie inside the intersection of these three triangles, the singleton $\{G\}$. The result follows.

(b) *Solution 1.* Suppose the vertices of the triangle are given by the planar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; the centroid of the triangle is at $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Suppose that one ant is placed at $t\mathbf{a} + (1-t)\mathbf{b}$ for $0 \leq t \leq 1$. Place the other two ants at $t\mathbf{b} + (1-t)\mathbf{c}$ and $t\mathbf{c} + (1-t)\mathbf{a}$. The centroid of the ants' triangle is at

$$\frac{1}{3}[(t\mathbf{a} + (1-t)\mathbf{b}) + (t\mathbf{b} + (1-t)\mathbf{c}) + (t\mathbf{c} + (1-t)\mathbf{a}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) \ .$$

(b) *Solution 2.* If one ant is at a vertex, then we can replace the remaining ants at the other vertices of the fixed triangle. Suppose, wolog, the ant is at X in the side BC.

Let MN be the line joining the midpoints M and N of AB and AC respectively; $MN \| BC$. Let XG meet MN at W. Since $BG : BN (= CG : CM) = 2 : 3$, it follows, by considering the similar triangles BGX and NGW, that $XG : XW = 2 : 3$. Hence the midpoint of the segment joining the other two ants' positions must be at W. Thus, the problem now is to find points Y and Z on the perimeter of ΔABC such that W is the midpoint of YZ. We use a continuity argument.

Let UV be any segment containing W whose endpoints lie on the perimeter of ΔABC. Let Y travel counterclockwise around the perimeter from U to V, and let Z be a point on the perimeter such that W lies on YZ. When Y is at U, $YW : WZ = VW : WV$, while when Y is at V, $YW : WZ = VW : WU$. Hence $YW : WZ$ varies continuously from a certain ratio to its reciprocal, so there must be a position for which $YW = WZ$.

(b) *Solution 3.* [A. Panayotov] Suppose that the triangle has vertices at $(0,0)$, $(1,0)$ and (u,v), so that its centroid is at $(\frac{1}{3}(1+u), \frac{v}{3})$. Wolog, let one ant be at $(a,0)$ where $0 \leq a \leq 1$. Put the second ant at (u,v). Then we will place the third ant at a point $(b,0)$ on the $x-$axis. We require that $\frac{1}{3}(a+b+u) = \frac{1}{3}(1+u)$, so that $b = 1-a$. Clearly, $0 \leq b \leq 1$ and the result follows.

656. Let ABC be a triangle and k be a real constant. Determine the locus of a point M in the plane of the triangle for which

$$|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k \ .$$

Solution. Let O and R be the circumcentre and circumradius, respectively, of triangle ABC. We have that

$$\begin{aligned}|MA|^2 = |\overrightarrow{MA}|^2 &= |\overrightarrow{MO} + \overrightarrow{OA}|^2 \\ &= |\overrightarrow{MO}|^2 + |\overrightarrow{OA}|^2 + 2\overrightarrow{MO} \cdot \overrightarrow{OA} \\ &= |\overrightarrow{MO}|^2 + R^2 + 2\overrightarrow{MO} \cdot \overrightarrow{OA}\end{aligned}$$

with similar expressions for MB and MC. Therefore, we have that

$$\begin{aligned}|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = (|MO|^2 + R^2)(\sin 2A + \sin 2B + \sin 2C) \\ 2\overrightarrow{MO} \cdot (\overrightarrow{OA} \sin 2A + \overrightarrow{OB} \sin 2B + \overrightarrow{OC} \sin 2C) \ .\end{aligned}$$

Now

$$\begin{aligned}\sin 2A + \sin 2B + \sin 2C &= \sin 2A + \sin 2B - \sin(2A + 2B) \\ &= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \\ &= 2 \sin A \cos A(2 \sin^2 B) + 2 \sin B \cos B(2 \sin^2 A) \\ &= 4 \sin A \sin B \sin(A + B) = 4 \sin A \sin B \sin C \\ &= \frac{2[ABC]}{R^2} \ ,\end{aligned}$$

since $[ABC] = \frac{1}{2}ab \sin C = 2R^2 \sin A \sin B \sin C$.

Also, we have that

$$\overrightarrow{OA} \sin 2A + \overrightarrow{OB} \sin 2B + \overrightarrow{OC} \sin 2C = \overrightarrow{O} \ .$$

To see this, let P be the intersection of the line AO with the side BC of the triangle. Observe that $\angle BOP = 180^\circ - 2\angle ACB$, $\angle COP = 180^\circ - 2\angle ABC$, $\angle OBC = \angle OCB = 90^\circ - \angle BAC$. Applying the Law of Sines to triangle OPC yields that

$$\frac{|OP|}{\sin(90^\circ - A)} = \frac{|OC|}{\sin(2C + A - 90^\circ)} \ .$$

Since $|OC| = R$, we find that

$$\begin{aligned}|OA| &= \frac{-\cos(2C + A)}{\cos A}|OP| = \frac{-2 \sin A \cos(2C + A)}{2 \sin A \cos A}|OP| \\ &= \frac{\sin 2B + \sin 2C}{\sin 2A}|OP| \ ,\end{aligned}$$

so that

$$\overrightarrow{OA} = -\frac{\sin 2B + \sin 2C}{\sin 2A}\overrightarrow{OP} \ .$$

Applying the Law of Sines in triangle BOP and COP, we obtain that

$$\frac{|OP|}{\sin(90^\circ - A)} = \frac{|BP|}{\sin 2C}$$

and

$$\frac{|OP|}{\sin(90^\circ - A)} = \frac{|CP|}{\sin 2B} \ .$$

Therefore $|BP| \sin 2B = |CP| \sin 2C$, so that

$$\sin 2B\overrightarrow{PB} = -\sin 2C\overrightarrow{PC}$$

and

$$\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = -(\sin 2B + \sin 2C)\overrightarrow{OP} + \sin 2B\overrightarrow{OB} + \sin 2C\overrightarrow{OC}$$
$$= \sin 2B(\overrightarrow{OB} - \overrightarrow{OP}) + \sin 2C(\overrightarrow{OC} - \overrightarrow{OP})$$
$$= \sin 2B\overrightarrow{PB} + \sin 2C\overrightarrow{PC} = \overrightarrow{O} \ .$$

Therefore $(|MO|^2 + R^2)(2[ABC]/R^2) = k$ so that

$$|MO|^2 = \frac{k - 2[ABC]}{2[ABC]}R^2 \ .$$

Therefore, when $k < 2[ABC]$, the locus is the empty set. When $k = 2[ABC]$, the locus consists solely of the circumcentre. When $k > 2[ABC]$, the locus is a circle concentric with the circumcircle.

657. Let a, b, c be positive real numbers for which $a + b + c = abc$. Find the minimum value of

$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \ .$$

Solution 1. By repeated squaring it can be shown that

$$\sqrt{x^2 + u^2} + \sqrt{y^2 + b^2} \geq \sqrt{(x + u)^2 + (y + v)^2} \ ,$$

for $x, y, u, v \geq 0$. Applying this inequality yields that

$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \geq \sqrt{(1 + 1)^2 + (\frac{1}{a} + \frac{1}{b})^2} + \sqrt{1 + \frac{1}{c^2}}$$
$$\geq \sqrt{(2 + 1)^2 + (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2} \ .$$

The given condition implies that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$, whereupon

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \geq 2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 2 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 3 \ .$$

It follows that the given expression is not less than $2\sqrt{3}$, with equality occurring if and only if $a = b = c = \sqrt{3}$.

Solution 2. [S. Sun] Using the inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$ for real x, y, z, we find that the square of the quantity in question is not less than

$$3\left(\sqrt{1 + \frac{1}{a^2}}\sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{b^2}}\sqrt{1 + \frac{1}{c^2}} + \sqrt{1 + \frac{1}{c^2}}\sqrt{1 + \frac{1}{a^2}}\right) \ .$$

From the Arithmetic-Geometric Means Inequality, we find that

$$\sqrt{1 + \frac{1}{a^2}}\sqrt{1 + \frac{1}{b^2}} = \sqrt{1 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{a^2b^2}} \geq \sqrt{1 + \frac{2}{ab} + \frac{1}{a^2b^2}} = 1 + \frac{1}{ab} \ ,$$

with similar inequalities for the other products. Since

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a + b + c}{abc} = 1 \ ,$$

we find that the square of the quantity in question is not less than $3 \times 4 = 12$, so that the quantity has the minimum value $2\sqrt{3}$, attainable if and only is $a = b = c = \sqrt{3}$.

Solution 3. Let A, B, C be acute angles for which $a = \tan A$, $b = \tan B$ and $c = \tan C$. Then

$$\begin{aligned} c &= -\frac{a+b}{1-ab} = -\frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= -\tan(A+B) = \tan(\pi - A - B) \ , \end{aligned}$$

so that $C = \pi - A - B$. Substituting these values fo a, b, c into the given expression yields

$$\csc A + \csc B + \csc C$$

. Since the cosecant function is convex in the interval $(0, \pi/2)$, by Jensen's inequality, we deduce that

$$\csc A + \csc B + \csc C \geq 3 \csc \left(\frac{A+B+C}{3} \right) = 3 \csc \frac{\pi}{3} = 2\sqrt{3} \ ,$$

with equality if and only if $A = B = C = \frac{\pi}{3}$. Thus, the minimum of the given expression is equal to $2\sqrt{3}$ with equality if and only is $a = b = c = \sqrt{3}$.

658. Prove that $\tan 20° + 4 \sin 20° = \sqrt{3}$.

Solution 1. [CJ. Bao] Since

$$(\sqrt{3}/2) \cos 20° - (1/2) \sin 20° = \sin 60° \cos 20° - \cos 60° \sin 20° = \sin 40° = 2 \sin 20° \cos 20° \ ,$$

it follows that

$$\sqrt{3} \cos 20° = \sin 20° + 4 \sin 20° \cos 20° a \ .$$

Division by $\cos 20°$ yields the desired result.

Solution 2. Let ABC be a triangle with $\angle ABC = 60°$ and $\angle CAB = 30°$. Let ABD be a triangle on the same side of AB with $\angle ABD = 40°$ and $\angle DAB = 50°$. Suppose that AC and BD intersect at E, and that the length of BC is 1, so that the respective lengths of CA and AB are $\sqrt{3}$ and 2. Then

$$|AD| = |AB| \sin 40° = 4 \sin 20° \cos 20°$$

and

$$|AE| = |AD| \sec 20° = |AB| \cos 50° \sec 20° = 2 \sin 40° \sec 20° = 4 \sin 20° \ .$$

However, $|CE| = |BC| \tan 20° = \tan 20°$. Therefore

$$\tan 20° + 4 \sin 20° = |CE| + |AE| = |AC| = \sqrt{3} \ .$$

Solution 3. [M. Essafty]

$$\begin{aligned} \tan 20° + 4 \sin 20° &= \frac{\sin 20° + 4 \sin 20° \cos 20°}{\cos 20°} \\ &= \frac{\sin 20° + 2 \sin 40°}{\cos 20°} \\ &= \frac{\sin(30° - 10°) + 2 \sin(30° + 10°}{\cos(30° - 10°} \\ &= \frac{3 \sin 30° \cos 10° + \sin 10° \cos 30°}{\cos 30° \cos 10° + \sin 30° \sin 10°} \\ &= \frac{3 \cos 10° + \sqrt{3} \sin 10°}{\sqrt{3} \cos 10° + \sin 10°} = \sqrt{3} \ . \end{aligned}$$

Solution 4.

$$\begin{aligned}\tan 20^\circ + 4\sin 20^\circ &= \frac{\sin 20^\circ + 4\sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\ &= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ} = \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ} \\ &= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3}\ .\end{aligned}$$

Solution 5.

$$\begin{aligned}\tan 20^\circ + 4\sin 20^\circ &= \frac{\sin 20^\circ + 4\sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\ &= \frac{\sin 50^\circ \cos 30^\circ - (1/2)\cos 50^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\ &= \frac{\sin 50^\circ \cos 30^\circ + (1/2)\cos 50^\circ + \cos 50^\circ}{\cos 20^\circ} \\ &= \frac{\sin 80^\circ + \cos 50^\circ}{\cos 20^\circ} = \frac{\cos 10^\circ + \cos 50^\circ}{\cos 20^\circ} \\ & \frac{2\cos 30^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3}\ .\end{aligned}$$

Solution 6. Let $a = \cos 20^\circ$. Then, using the de Moivre formula $\cos 3\theta + i \sin 3\theta = (\cos\theta + i\sin\theta)^3$ with $\theta = 20^\circ$, we find that

$$\frac{1}{2} = \cos 60^\circ = 4a^3 - 3a$$

and

$$\frac{\sqrt{3}}{2} = 3\sin 20^\circ - 4\sin^3 20^\circ = \sin 20^\circ(3 - 4(1-a^2)) = \sin 20^\circ(4a^2 - 1)\ .$$

Therefore

$$\tan 20^\circ + 4\sin 20^\circ - \sqrt{3} = \sin 20^\circ[(1/a) + 4 - 8a^2 + 2) = a^{-1}\sin 20^\circ(1 + 6a - 8a^3) = 0\ .$$

Solution 7. [B. Wu]

$$\begin{aligned}\tan 60^\circ - \tan 20^\circ &= \frac{\sin 60^\circ}{\cos 60^\circ} - \frac{\sin 20^\circ}{\cos 20^\circ} \\ &= \frac{\sin 40^\circ}{\cos 60^\circ \cos 20^\circ} = 4\sin 20^\circ \cos 40^\circ over \cos 20^\circ = 4\sin 20^\circ\ ,\end{aligned}$$

whence $\tan 20^\circ + 4\sin 20^\circ = \sqrt{3}$.

659. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a-1$, $2b-1$ and $a+b$ are all primes. Determine all possibilities for which a and b are themselves prime.

(b) Suppose a and b are positive integers such that $2a-1$, $2b-1$ and $a+b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a+b$.

(a) *First solution.* $(a, b) = (3, 2)$ yields $2a-1 = 5$, $2b-1 = 3$ and $a+b = 5$; $(a, b) = (3, 4)$ yields $2a-1 = 5$, $2b-1 = 7$ and $a+b = 7$. Suppose that a and b are primes. Then for $a+b$ to be prime, $a+b$ must be odd, so that one of a and b, say b, is equal to 2. Thus, we require the $a+2$ and $2a-1$, along with a, to be prime. This is true when $a = 3$.

Now suppose a is an odd prime exceeding 3. Then $a \equiv \pm 1 \pmod 6$, so the only way a and $a+2$ can both be prime is for $a \equiv -1 \pmod 6$, whence $2a - 1 \equiv -3 \pmod 6$. Thus, 3 divides $2a-1$, and since $2a - 1 \geq 9$, $2a-1$ must be composite.

(b) *Solution 1.* We first recall a bit of theory. Let p be a prime. By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod p$ whenever $\gcd(a,p) = 1$. Let d be the smallest positive integer for which $a^d \equiv \pm 1 \pmod p$. Then d divides $p-1$, and indeed divides any positive integer k for which $a^k \equiv \pm 1 \pmod p$. Now to the problem.

Since $a+b$ is prime, $a \neq b$. Wolog, let $a > b$ and let $p = a+b$. Then $a \equiv -b \pmod p$, so that

$$a^b + b^a \equiv (-b)^b + b^a \equiv b^b((-1)^b + b^{a-b}) \ .$$

Suppose, if possible, that p divides $a^b + b^a$. Then, since $b < p$, $\gcd(b,p) = 1$ and so $b^{a-b} \equiv (-1)^{b+1} \pmod p$. It follows that

$$b^{2b-1} = b^{(p-1)-(a-b)} \equiv (-1)^{b+1} \mod p \ .$$

Now $2b-1$ is prime, so that $2b-1$ must be the smallest exponent d for which $b^d \equiv \pm 1 \pmod p$. Hence $2b-1$ divides $a-b$, so that for some positive integer c, $a - b = c(2b-1)$, whence $a = b + 2bc - c$ and so

$$2a - 1 = 2b - 1 + (2b-1)2c = (2b-1)(2c+1) \ .$$

But $2a-1$ is prime and $2b-1 > 1$, so $2c+1 = 1$ and $c = 0$. This is a contradiction. Hence p does not divide $a^b + b^a$.

Similarly, using the fact that $a^b + b^a \equiv (-b)^a + b^b \equiv b^b((-1)^a b^{a-b} + 1)$, we can show that p does not divide $a^a + b^b$.

(b) *Solution 2.* [M. Boase] Suppose that a and b exist as specified. Exactly one of a and b is odd, since $a+b$ is prime. Let it be a. Modulo $a+b$, we have that

$$0 \equiv a^b + b^a = a^b + (-a)^a \equiv a^b - a^a \equiv a^a(a^{b-a} - 1) \text{ or } a^b(1 - a^{a-b})$$

according as $a < b$ or $a > b$. Hence $a^{|b-a|} - 1 \equiv 0 \pmod{a+b}$. Now $a + b - 1 \pm |b-a| = 2a-1$ or $2b-1$, and $a^{a+b-1} \equiv 1 \pmod{a+b}$ (by Fermat's Little Theorem). Hence $a^{2a-1} \equiv a^{2b-1} \equiv 1 \pmod{a+b}$. Both $2a-1$ and $2b-1$ exceed 1 and are divisible by the smallest value of m for which $a^m \equiv 1 \pmod{a+b}$. Since both are prime, $2a-1 = 2b-1 = m$, whence $a = b$, a contradiction. A similar argument can be applied to $a^a + b^b$.

(c) *Solution 3.* Suppose, if possible, that one of $a^b + b^a$ and $a^a + b^b$ is divisible by $a+b$. Then $a+b$ divides their product $a^{a+b} + (ab)^a + (ab)^b + b^{a+b}$. By Fermat's Little Theorem, $a^{a+b} + b^{a+b} \equiv a + b \equiv 0 \pmod{a+b}$, so that $(ab)^a + (ab)^b \equiv 0 \pmod{a+b}$. Since $a+b$ is prime, it is odd and so $a \neq b$. Wolog, let $a > b$. Then

$$(ab)^a + (ab)^b = (ab)^b[(ab)^{a-b} + 1]$$

and $\gcd(a, a+b) = \gcd(b, a+b) = 1$, so that $(ab)^{a-b} + 1 \equiv 0 \pmod{a+b}$. Since $(ab)^{a+b-1} \equiv 1 \pmod{a+b}$, it follows that $(ab)^{2a-1} \equiv (ab)^{2b-1} \equiv -1 \pmod{a+b}$. As in the foregoing solution, it follows that $a = b$, and we get a contradiction.

660. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.

Solution 1. Let the lengths a, b, c, u and the angles α, β, γ, λ, μ, ν be as indicated in the diagram.

In the solution, we make use of the fact that if $p/q = r/s$, then both fractions are equal to $(p+r)/(q+s)$. Since $\angle DBP = 90^\circ + \lambda - 2\beta$, it follows that

$$2\mu = 180^\circ - (90^\circ - \alpha) - (90^\circ + \lambda - 2\beta) = \alpha + 2\beta - \lambda \ .$$

Similarly, $2\nu = \alpha + 2\gamma - \lambda$. Using the Law of Sines, we find that

$$\frac{a}{\sin 2\alpha} = \frac{b}{\sin 2\beta} = \frac{c}{\sin 2\gamma} = \frac{b+c}{\sin 2\beta + \sin 2\gamma} = \frac{b+c}{2\sin(\beta+\gamma)\cos(\beta-\gamma)}$$
$$= \frac{b+c}{2\cos\alpha\cos(\beta-\gamma)} \ .$$

Hence

$$\frac{a}{\sin\alpha} = \frac{b+c}{\cos(\beta-\gamma)} \ .$$

Since $a = 2u\sin\lambda$ and, by the Law of Sines,

$$\frac{u}{\sin(90^\circ-\alpha)} = \frac{b}{\sin 2\mu} \quad \text{and} \quad \frac{u}{\sin(90^\circ-\alpha)} = \frac{c}{\sin 2\nu} \ ,$$

we have that

$$\frac{a}{2\sin\lambda\cos\alpha} = \frac{u}{\cos\alpha} = \frac{b}{\sin 2\mu} = \frac{c}{\sin 2\nu} = \frac{b+c}{\sin 2\mu + \sin 2\nu}$$
$$= \frac{b+c}{2\sin(\mu+\nu)\cos(\mu-\nu)} = \frac{b+c}{2\cos\lambda\cos(\beta-\gamma)} = \frac{a}{2\cos\lambda\sin\alpha} \ .$$

Hence $\tan\alpha = \tan\lambda$ and so $\alpha = \lambda$.

Solution 2. Let M be the midpoint of BC. A rotation of 180° about M interchanges B and C and takes E to G, D to F and P to Q. Then $AB = CE = BG$ and $AC = BD = CF$. Join GA and FA. Let $2\alpha = \angle BAC$. Since $AE \| BG$ and AB is a transversal, $\angle GBA = \angle BAC = 2\alpha$. Since $AB = BG$, $\angle BGA = 90^\circ - \alpha$. But $\angle BGF = \angle CED = 90^\circ - \alpha$. Thus, G, A, F are collinear.

Since GF and DE are equidistant from M, we can use Cartesian coordinates with the origin at M, the line $y = 1$ as GF and the line $y = -1$ as DE. Let $A \sim (a, 1)$, $B \sim (-u, -mu)$, $C \sim (u, mu)$. Then $P \sim (m, -1)$, $Q \sim (-m, 1)$,

$$D \sim (a - \frac{2(a+u)}{1+mu}, -1), \qquad E \sim (a + \frac{2(a+u)}{1+mu}, -1) \ .$$

Since $|AC| = |BD|$, we find that $u - a = -u - a + \frac{2(a+u)}{1+mu}$, or $a = mu^2$. (We can check this by equating the slopes of AC and AE.)

The slope of AE is $-1/u$ and of AD is $1/u$, so that

$$\tan\angle BAC = \frac{-(2/u)}{1-(1/u^2)} = -\frac{2u}{u^2-1} \ .$$

The slope of CQ is $(mu-1)/(m+u)$ and of BQ is $(1+mu)/(u-m)$, so that

$$\tan\angle BPC = \tan\angle BQC = \frac{(mu-1)(u-m)-(mu+1)(u+m)}{(u-m)(u+m)+(mu-1)(mu+1)}$$
$$= \frac{-2(m^2u+u)}{u^2-m^2+m^2u^2-1} = \frac{-2(m^2+1)u}{(1+m^2)(u^2-1)} = \frac{-2u}{u^2-1} \ .$$

The result follows.

Solution 3. [M. Boase] Let XAY be drawn parallel to DE.

Since M is the midpoint of BC, the distance from M to DE is the average of the distances from B and C to DE. Similarly, the distance from M to XY is the average of the distances from B and C to XY. The distance of B (resp. C) to DE equals the distance of C (resp. B) to XY. Hence, M is equidistant from DE and XY. If PM produced meets XY in Q, then $PM = MQ$ and so $\angle BQC = \angle BPC$.

Select R on MQ (possibly produced) so that $\angle BAC = \angle BRC$. Since $\Delta ADE \| |\Delta RBC$, $\angle RBC = \angle RCB = \angle ADE$. Since $BARC$ is a concyclic quadrilateral, $\angle BAR = 180° - \angle RCB = 180° - \angle ADE = 180° - \angle XAD = \angle BAQ$ from which it follows that $R = Q$ and so $\angle BPC = \angle BQC = \angle BRC = \angle BAC$.

Solution 4. [Jimmy Chui] Set coordinates: $A \sim (0, (m+n)b)$, $B \sim (-ma, nb)$, $C \sim (na, mb)$ $D \sim (-(m+n)a, 0)$ and $E \sim ((m+n)a, 0)$ where $m = |AB|$, $n = |AC|$ and $a^2 + b^2 = 1$. Then the line BC has the equation

$$\frac{m-n}{a}x - \frac{m+n}{b}y + m^2 + n^2 = 0$$

and the right bisector of BC has equation

$$\frac{m+n}{b}x + \frac{m-n}{a}y + \frac{(a^2-b^2)(m^2-n^2)}{2ab} = 0 \quad .$$

Thus

$$P \sim \left(\frac{(b^2-a^2)(m-n)}{2a}, 0 \right) \; .$$

Now

$$|BC|^2 = m^2 + n^2 + 2mn(a^2 - b^2)$$

and

$$|BP|^2 = \frac{m^2 + n^2 + 2mn(a^2-b^2)}{4a^2}$$

so that $|BC|/|BP| = 2a$. Also $|DE|/|AD| = 2(m+n)a/(m+n) = 2a$ so that ΔBPC is similar to ΔADE and the result follows.

Solution 5. Determine points L and N on DE such that $BL \| AE$ and $LN = NE$. Now

$$\frac{LE}{LD} = \frac{AB}{BD} = \frac{CE}{CA}$$

so that $CL \| AD$ and $CL : AD = CE : AE$. Since $AD = DE$, $CL = CE$ and so $CN \perp LE$. Consider the trapezoid $CBLE$. The line MN joins the midpoints of the nonparallel opposite sides and so $MN \| BL$. $MPNC$ is a quadrilateral with right angles at M and N, and so is concyclic. Hence

$$\angle BPC = 2\angle MPC = 2\angle MNC = 2\angle NCE = \angle LCE = \angle BAC \quad .$$

Solution 6. [C. So] Let F, N, G be the feet of the perpendiculars dropped from B, M, C respectively to DE. Note that $FN = NG$, so that $MF = MG$. Let $\angle ADE = \angle AED = \theta$, $|AB| = c$, $|AC| = b$ and h be the altitude of ΔADE. Then

$$|MN| = \frac{1}{2}[|BF| + |CG|] = \frac{1}{2}(b+c)\sin\theta = \frac{h}{2}$$

and

$$|DF| = b\cos\theta \;, \quad |GE| = c\cos\theta \;, \quad |DE| = 2(b+c)\cos\theta \quad .$$

Hence $|FG| = |DE| - |DF| - |GE| = \frac{1}{2}|DE|$. Since ΔADE and ΔMFG are isosceles triangles with heights and beses in proportion, they are similar so that $\angle MFG = \angle ADE = \theta$. Since $\angle BFP = \angle BMG = 90°$, the quadrilateral $BFPM$ is concyclic and so $\angle CBP = \angle MFP = \theta$ (we are supposing that the configuration is labelled so P lies between F and E). Hence ΔADE is similar to ΔPCB and so $\angle BPC = \angle BAC$.

Solution 7. [A. Chan] Let $\angle ADE = \angle AED = \theta$, so $\angle BAC = 180° - 2\theta$. Suppose that $\angle ACB = \phi$, $\angle CPE = \sigma$ and $\angle BCP = \rho$. By the Law of Sines for triangles ABC and PCE, we find that

$$\frac{2|PC|\cos\rho}{\sin 2\theta} = \frac{|AB|}{\sin\phi}$$

whence

$$\frac{\sin\sigma}{\sin\theta} = \frac{|CE|}{|PC|} = \frac{|AB|}{|PC|} = \frac{2\cos\rho\sin\phi}{\sin 2\theta}$$

and

$$\sin\sigma\cos\theta = \sin\phi\cos\rho \quad .$$

Therefore

$$\sin(\theta+\sigma) + \sin(\sigma-\theta) = \sin(\phi+\rho) + \sin(\phi-\rho).$$

Since $\theta+\sigma = \phi+\rho$, $\sin(\sigma-\theta) = \sin(\phi-\rho)$. Either $(\sigma-\theta)+(\phi-\rho) = \pm 180°$ or $\sigma-\theta = \phi-\rho$. In the first case, since $\theta+\sigma = \phi+\rho$, $|\sigma-\rho| = 90°$, which is false.

Hence $\sigma-\theta = \phi-\rho$, so, with $\theta+\sigma = \phi+\rho$, we have that

$$2\theta = \theta + (\rho+\sigma-\phi) = \theta + (\rho+\rho-\sigma) = 2\rho$$

and the result follows.

Solution 8. [A. Murali] Let F be the midpoint of BC. Observe that triangles ADE and PBC are isosceles with $AD = AE$ and $PB = PC$. Suppose that the line parallel to AC through D and the line parallel to AD through C meet at N, and let CN intersect DE at M. Since $ACND$ is a parallelogram, $DN = AC$. Since triangle CME is similar to triangle ADE, it is isosceles with $CM = CE = AB$. Since $AD = CN$, $BMND$ is a parallelogram. In fact, $MN = BD = AC = DN = BM$, so that $BMND$ is a rhombus.

Since P is a point on a diagonal of the rhombus $BMND$, $PB = PN$ and so triangles PBM and PNM are congrunent, from which we see that $\angle PBM = \angle PNM$. Since $PC = PB = PN$, it follows that $\angle PBM = \angle PNC = \angle PCM$ and quadrilateral $BCMP$ is concyclic. Therefore, $\angle BPC = \angle BMC = \angle BAC$ ($ABMC$ being a quadrilateral).

Solution 9. [C. Deng] If BC were parallel to DE, then BC would be a midline of triangle ADE and P would be the reflection of A in the axis BC yielding the desired result. Suppose that BC and DE are not parallel. Let R be the circumradius of triangle ADE, R_1 the circumradius of triangle BDP and R_2 the circumradius of triangle CEP. Observe that $AD = AE$ and $PB = PC$.

Let the circumcircles of triangles BDP and CEP intersect at O. The point O lies inside triangle ADE. By the Extended Sine Law,

$$\frac{OP}{\sin\angle PBO} = 2R_1 = \frac{PB}{\sin\angle ADE} = \frac{PC}{\sin\angle AED} = 2R_2 = \frac{OP}{\sin\angle PCO} \ .$$

Since $\angle PCO = \angle PEO < \angle PEA < 90°$, the angle PCO is acute. Similarly, angle PBO is acute. Therefore $\angle PBO = \angle PCO$, so that $\angle OBC = \angle OCB$ and O is on the right bisector of BC. Since

$$DO = 2R_1 \sin\angle DPO = 2R_2\sin\angle OPE = EO$$

, the point O is on the right bisector of DE, which is also the angle bisector of $\angle BAC$.

Since the quadrilaterals $OBDP$ and $OCEP$ are concyclic,

$$\begin{aligned}\angle BOC &= 360° - \angle BOP - \angle COP \\ &= 36° - (180° - \angle BDP) - (180° - \angle CEP) \\ &= \angle ADE + \angle AED = 180° - \angle BAC \ .\end{aligned}$$

Hence quadrilaterla $ABOC$ is concyclic. Also $\angle BCO = \angle CBO = \frac{1}{2}\angle BAC$.

From Ptolemy's Theorem, we have that

$$BC \cdot AO = AB \cdot CO + AC \cdot BO = (AB + AB \cdot BO = AD \cdot BO \ .$$

Therefore

$$AO = AD \cdot \frac{BO}{BC} = AD \cdot \frac{\sin \angle BCO}{\sin BOC} = AD \cdot \frac{\sin \frac{1}{2}\angle BAC}{2 \sin \angle BAC} = \frac{AD}{2 \cos \frac{1}{2}\angle BAC} = R \ .$$

Since O is on the right bisector of DE and $AO = R$, O is the circumcentre of triangle ADE. Therefore

$$\angle BPC = \angle BPO + \angle CPO = \angle BDO + \angle CEO = \angle OAB + \angle OAC = \angle A \ .$$