Solutions for February

585. Calculate the number
$$b = \lfloor (\sqrt{n-1} + \sqrt{n} + \sqrt{n+1})^2 \rfloor \ .$$

Solution. When $n = 1$, then $b = \lfloor 3 + 2\sqrt{3} \rfloor = 5$. When $n = 2$, then
$$4.14 = 1 + 1.41 + 1.73 < 1 + \sqrt{2} + \sqrt{3} < 1 + \frac{17}{12} + \frac{7}{4} = \frac{25}{6}$$
so that $17 < (1 + \sqrt{2} + \sqrt{3})^2 < 18$ and $\lfloor (1 + \sqrt{2} + \sqrt{3})^2 \rfloor = 17 = 9 \times 2 - 1$.

Let $n \geq 3$. Since
$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}} = \sqrt{n} - \sqrt{n-1} \ ,$$
it follows that $\sqrt{n-1} + \sqrt{n+1} < 2\sqrt{n}$. Also
$$\sqrt{n-1} + \sqrt{n} + \sqrt{n+1} > 3\sqrt[3]{n^3 - n} > \sqrt{9n-1} \ ;$$
the first inequality follows from that of the arithmetic and geometric means, and the second is evident after raising to the sixth power. Therefore $\sqrt{9n-1} < \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n}$.

Hence
$$9n - 1 < (\sqrt{n-1} + \sqrt{n} + \sqrt{n+1})^2 < 9n \ ,$$
and so $b = 9n - 1$, when $n \geq 2$.

598. Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive integers. If possible, select two indices j, k with $1 \leq j < k \leq n$ for which a_j does not divide a_k; replace a_j by the greatest common divisor of a_j and a_k, and replace a_k by the least common multiple of a_j and a_k. Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

Solution. Let $\{p_i : 1 \leq i \leq m\}$ be the set of of all primes, listed in some order, dividing at least one of the a_i. All the terms of any sequence thereafter are divisible by only these primes. For each sequence obtained and for each prime p_i, define a vector with n components whose sth entry is the exponent of the highest power of p_i that divides the sth term of the sequence.

Suppose that $a_j = \prod_{s=1}^m p_s^{u_s}$ and $a_k = \prod_{s=1}^m p_s^{v_s}$ are two terms of one of the sequences. Then gcd $(a_j, a_k) = \prod_{s=1}^m p_s^{w_s}$ and lcm $(a_j, a_k) = \prod_{s=1}^m p_s^{z_s}$, where w_s is the minimum and z_s is the maximum of u_s and v_s for each s. The condition that a_j divides a_k is equivalent to $u_s \leq v_s$ for each s.

Let us see what the effect of the operation on a sequence has on the m vectors associated with the sequence. If two elements, the jth and kth for which the jth does not divide the kth, then there is at least one vector for which the jth term is larger than the kth term. The operation just interchanges these terms. This reduces the number of pairs of components of the vector for which the earlier one exceeds the second.

Since there are only finitely many vectors (one for each prime) and each vector has only finitely many component pairs, the process must terminate after a finite number of operations. No moves are possible only when each vector is increasing. Since each move permutes the entries of each vectors, in the final stage we must obtain the unique rearrangement of each vector in which the components are increasing. The kth terms of the vectors give the exponents of the primes p_s that constitute the prime factorization of the kth term of the sequence at the end. The result follows.

599. Determine the number of distinct solutions x with $0 \leq x \leq \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8\cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

Solution 1. (a) It is clear that no multiple of π satisfies the equation. So we must have that $\sin x \neq 0$. Multiply the equation by $\sin x$ to obtain

$$8\sin x \cos x \cos 2x \cos 4x = 4 \sin 2x \cos 2x \cos 4x = 2 \sin 4x \cos 4x = \sin 8x \ .$$

Hence the given equation is equivalent to $\sin 8x = \sin x$ with $\sin x \neq 0$. Hence, we must have $x + 8x = (2k+1)\pi$, $8x = (2k)\pi + x$, since $0 \leq x \leq \pi$. These lead to $x = \pi/9$ ($20°$), $x = 2\pi/7$, $x = \pi/3$ ($60°$), $x = 4\pi/7$, $x = 5\pi/9$ ($100°$), $x = 6\pi/7$ ($120°$), $x = 7\pi/9$. Thus there are seven solutions to the equation.

(b) [Z. Liu] It can be checked that no multiple of π nor any odd multiple of $\pi/4$ satisfies the equation. The truth of the equation implies that

$$\begin{aligned} \sin 8x \cos 5x &= 2\sin 4x \cos 4x \cos 5x = 4 \sin 2x \cos 2x \cos 4x \sin 5x \\ &= (\sin x \cos 2x)(8 \cos x \cos 4x \cos 5x) = \sin x \cos 2x \ . \end{aligned}$$

Using the product to sum conversion formula yields

$$\sin 13x + \sin 3x = \sin 3x - \sin x \ ,$$

whence $\sin 13x = \sin(-x)$. Therefore, either $12x = 13x + (-x)$ is an odd multiple of π or $14x = 13x - (-x)$ is an even multiple of π. However, $x = 0, \pi/4, \pi/2, 3\pi/4$ are extraneous solutions that do not satisfies the given equation. Therefore, there are ten solutions, namely

$$x = \frac{\pi}{12}, \frac{5\pi}{12}. \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7} \ .$$

Solution 2. (a) Let $t = \cos x$. Then $\cos 2x = 2t^2 - 1$ and $\cos 4x = 2(2t^2-1) - 1 = 8t^4 - 8t^2 + 1$, so that

$$\cos x \cos 2x \cos 4x = t(2t^2 - 1)(8t^4 - 8t^2 + 1) \ .$$

Let

$$\begin{aligned} f(t) &= 8t(2t^2-1)(8t^4 - 8t^2 + 1) - 1 \\ &= 128t^7 - 192t^5 + 80t^3 - 8t - 1 \\ &= (2t-1)(64t^6 + 32t^5 - 80t^4 - 40t^3 + 20t^2 + 10t + 1) \\ &= (2t-1)(8t^3 + 4t^2 - 4t - 1)(8t^3 - 6t - 1) \ . \end{aligned}$$

(The factor $(2t-1)$ can be found by noting that $x = \pi/3$, corresponding to $t = 1/2$, is an obvious solution to the equation given in the problem.)

Let $g(t) = 8t^3 + 4t^2 - 4t - 1$ and $h(t) = 8t^3 - 6t - 1$. Since $g(-1) = -9$, $h(-1) = -1$, $g(-\frac{1}{2}) = h(-\frac{1}{2}) = 1$, $g(0) = h(0) = -1$, $g(1) = 7$ and $h(1) = 1$, both of $g(t)$ and $h(t)$ have a root in each of the intervals $(-1, -\frac{1}{2})$, $(-\frac{1}{2}, 0)$ and $(0, 1)$.

Since the only roots of $g(t) - h(t) = 4t^2 + 2t = 2t(2t+1)$ are $-\frac{1}{2}$ and 0, $g(t)$ and $h(t)$ do not have a root in common. Therefore , $f(t)$ has seven roots and these correspond to seven solutions of the given equation.

(b) We have that

$$\begin{aligned} 1 &= 8 \cos x \cos 4x \cos 5x = 4 \cos^2 4x + 4 \cos 4x \cos 6x \\ &= (2\cos 8x + 2) + (2 \cos 2x + 2 \cos 10x) \ , \end{aligned}$$

so that

$$2 \cos 2x + 2 \cos 8x + 2 \cos 10x + 1 = 0 \ .$$

Substituting $t = \cos 2x$ yields $\cos 4x = 2t^2 - 1$, $\cos 8x = 8t^2 - 4t^2 + 1$, $\cos 10x = 16t^5 - 20t^3 + 5t$, so that the equation becomes

$$0 = (4t^2 - 3)(8t^3 + 4t^2 - 4t - 1) \ .$$

The polynomial $4t^2 - 3$ has two roots in the interval $[-1, 1]$ corresponding to four values of x in the interval $[0, \pi]$. Let $f(t) = 8t^3 + 4t^2 - 4t - 1$. Since $f(-1) = -1$, $f(-\frac{1}{2}) = 1$, $f(0) = -1$, $f(1) = 7$, $f(t)$ has three real roots, once in each of the intervals $(-1, -\frac{1}{2})$, $(-\frac{1}{2}, 0)$, $(0, 1)$, and each of these corresponds to two solution x in the interval $[0, \pi]$. Therefore, the equation in x has ten solutions in the interval.

Comments. (a) The seven solutions of the equation $\sin 8x = \sin x$ can be seen from a sketch of the graphs of the two functions on the same axes.

(b) Since $2 \cos x \cos 5x = \cos 4x + \cos 6x$, the equation is equivalent to

$$4(\cos^2 4x + \cos 4x \cos 6x) = 1 \ .$$

Some solutions can be found by solving $\cos 6x = 0$ and $\cos^2 4x = \frac{1}{4}$. These are satisfied by $x = \pi/12$, $5\pi/12$, $7\pi/12$ and $11\pi/12$.

The trial, taking $\cos 4x = \frac{1}{2}$, is also reasonable, as it gives $x = \pi/12$. With this substitution, the left side become $4 \cos \pi/12 \sin \pi 12 = 2 \sin \pi/6 = 1$. The other multiples of $\pi/12$ can be handled in the same way.

When $t = \cos 2x$, there is another route to the equation in t to be analyzed. The equation, in the form, $1 = 4(\cos 4x)(\cos 4x + \cos 6x)$, is transformed to

$$1 = 4(2t^2 - 1)(2t^2 - 1 + 4t^3 - 3t) = 4(8t^5 + 4t^4 - 10t^2 - 4t^2 + 3t + 1) \ .$$

This simplifies to

$$\begin{aligned} 0 &= 32t^5 + 16t^4 - 40t^3 - 16t^2 + 12t + 3 \\ &= (4t^2 - 3)(8t^3 + 4t^2 - 4t - 1) \ . \end{aligned}$$

Since $x = \pi/12$ is a solution, $t = \cos \pi/6 = \sqrt{3}/2$ satisfies the equation in t and accounts for the factor $4t^2 - 3$ on the right side of the equation.

600. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n + b^n}{2}} \ .$$

Solution 1. Dividing the inequality through by $(b + a)/2$ yields the equivalent inequality

$$1 \leq \sqrt[n]{\frac{b'^{n+1} - a'^{n+1}}{(b' - a')(n+1)}} \leq \sqrt[n]{\frac{a'^n + b'^n}{2}} \ ,$$

with $a' = (2a)/(b + a)$ and $b' = (2b)/(b + a)$. Note that $(a' + b')/2 = 1$. and we can write $b' = 1 + u$ and $a' = 1 - u$ with $0 < u < 1$. The central term becomes the nth root of

$$\begin{aligned} \frac{(1+u)^{n+1} - (1-u)^{n+1}}{2(n+1)u} &= \frac{2[(n+1)u + \binom{n+1}{3}u^3 + \binom{n+1}{5}u^5 + \cdots]}{2(n+1)u} \\ &= 1 + \frac{1}{3}\binom{n}{2}u^2 + \frac{1}{5}\binom{n}{4}u^4 + \cdots \end{aligned}$$

which clearly exceeds 1 and gives the left inequality. The right term become the nth roots of

$$\frac{1}{2}[(1+u)^n + (1-u)^n] = 1 + \binom{n}{2}u^2 + \binom{n}{4}u^4 + \cdots$$

and the right inequality is true.

Solution 2. The inequality

$$\sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}} \leq \sqrt[n]{\frac{a^n+b^n}{2}}$$

is equivalent to

$$0 \leq (n+1)(a^n+b^n) - \frac{2(b^{n+1}-a^{n+1})}{b-a}.$$

The right side is equal to

$$\begin{aligned}
(n+1)(a^n+b^n) - 2(b^n + b^{n-1}a + b^{n-2}a^2 \cdots + b^2a^{n-2} + ba^{n-1} + a^n)& \\
= (a^n - b^n) + (a^n - b^{n-1}a) + (a^n - b^{n-2}a) + \cdots + (a^n - ba^{n-1}) + (a^n - a^n)& \\
+ (b^n - b^n) + (b^n - b^{n-1}a) + \cdots + (b^n - ba^{n-1}) + (b^n - a^n)& \\
= (a^n - b^n) + a(a^{n-1} - b^{n-1}) + a^2(a^{n-2} - b^{n-2}) + \cdots + a^{n-1}(a-b) + 0& \\
+ 0 + b^{n-1}(b-a) + \cdots + b(b^{n-1} - a^{n-1}) + (b^n - a^n)& \\
= 0 + (b-a)(b^{n-1} - a^{n-1}) + (b^2 - a^2)(b^{n-2} - a^{n-2}) + \cdots + (b^{n-1} - a^{n-1})(b-a)& \\
> 0\ .&
\end{aligned}$$

The left inequality

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$\left(\frac{b+a}{2}\right)^n \leq \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \quad .$$

Let $v = \frac{1}{2}(b-a)$ so that $b + a = 2(a + v)$. Then

$$\begin{aligned}
\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} - \left(\frac{b+a}{2}\right)^n &= \frac{(a+2v)^{n+1}-a^{n+1}}{(b-a)(n+1)} - (a+v)^n \\
&= \frac{1}{2v(n+1)}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n+1-k}(2v)^k\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n-(k-1)}(2v)^{k-1}\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=0}^{n}\binom{n+1}{k+1}a^{n-k}(2v)^k\right) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k \\
&= \left(\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}a^{n-k}(2v)^k - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k\right) \\
&= \sum_{k=0}^{n}\left(\frac{2^k}{k+1} - 1\right)\binom{n}{k}a^{n-k}v^k \geq 0 \quad ,
\end{aligned}$$

since $2^k = (1+1)^k = 1 + k + \binom{k}{2} + \cdots \geq 1 + k$ with equality if and only if $k = 0$ or 1. The result follows.

Solution 3. [D. Nicholson] (partial) Let $n \geq 2i + 1$. Then

$$(b^{n-i}a^i + b^ia^{n-i}) - (b^{n-i-1}a^{i+1} + b^{i+1}a^{n-i-1}) = (b-a)a^ib^i(b^{n-2i-1} - a^{n-2i-1}) \geq 0\ .$$

Hence, for $0 \le j \le \frac{1}{2}(n+1)$,

$$b^n + a^n \ge b^{n-1}a + ab^{n-1} \ge \cdots \ge b^{n-j}a^j + b^j a^{n-j} \quad .$$

When $n = 2k+1$,

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k}(b^{n-i}a^i + b^i a^{n-i}) \le (k+1)(b^n + a^n) = \frac{n+1}{2}(b^n + a^n)$$

and when $n = 2k$, we use the Arithmetic-Geometric Means Inequality to obtain $b^k a^k \le \frac{1}{2}(a^{2k} + b^{2k})$, so that

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k-1}(b^{n-i}a^i + b^i a^{n-i}) + b^k a^k \le k(b^n + a^n) + \frac{b^n + a^n}{2} = \frac{n+1}{2}(b^n + a^n) \ .$$

Hence

$$\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \le \frac{b^n + a^n}{2} \ .$$

Solution 4. [Y. Shen] Let $1 \le k \le n$ and $1 \le i \le k$. Then

$$(b^{k+1} + a^{k+1}) - (b^i a^{k+1-i} + a^i b^{k+1-i}) = (b^i - a^i)(b^{k+1-i} - a^{k+1-i}) \ge 0 \ .$$

Hence

$$k(b^{k+1} + a^{k+1}) \ge \sum_{i=1}^{k}(b^i a^{k+1-i} + a^i b^{k+1-i}) = 2\sum_{i=1}^{k} b^i a^{k+1-i} \quad .$$

This is equivalent to

$$\begin{aligned}
(2k+2)\sum_{i=0}^{k+1} b^i a^{k+1-i} &= (2k+2)(b^{k+1} + a^{k+1}) + (2k+2)\sum_{i=1}^{k} b^i a^{k+1-i} \\
&\ge (k+2)(b^{k+1} + a^{k+1}) + (2k+4)\sum_{i=1}^{k} b^i a^{k+1-i} \\
&= (k+2)(b^{k+1} + 2\sum_{i=1}^{k} b^i a^{k+1-i} + a^{k+1}) \\
&= (k+2)(b+a)\sum_{i=0}^{k} b^i a^{k-i}
\end{aligned}$$

which in turn is equivalent to

$$\frac{\sum_{i=0}^{k+1} b^i a^{k+1-i}}{k+2} \ge \frac{(b+a)(\sum_{i=0}^{k} b^i a^{k-i})}{2(k+1)} \quad .$$

We establish by induction that

$$\left(\frac{b+a}{2}\right)^n \le \frac{1}{n+1}\sum_{i=0}^{n} b^i a^{n-i}$$

which will yield the left inequality. This holds for $n = 1$. Suppose that it holds for $n = k$. Then

$$\begin{aligned}
\left(\frac{b+a}{2}\right) &= \left(\frac{b+a}{2}\right) \cdot \left(\frac{b+a}{2}\right)^k \\
&\le \left(\frac{b+a}{2}\right) \cdot \left(\frac{1}{k+1}\right)\sum_{i=0}^{k} b^i a^{k-i} \le \frac{1}{k+2}\sum_{i=0}^{k+1} b^i a^{k-i} \quad .
\end{aligned}$$

As above, we have, for $k = n - 1$,

$$(n-1)(b^n + a^n) \geq 2\sum_{i=1}^{n-1} b^i a^{n-i}$$

so that

$$(n+1)(b^n + a^n) \geq 2\sum_{i=0}^{n} b^i a^{n-i} = 2\left(\frac{b^{n+1} - a^{n+1}}{b-a}\right)$$

from which the right inequality follows.

Comment. The inequality

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$0 \leq \frac{2^n(b^{n+1} - a^{n+1})}{b-a} - (n+1)(b+a)^n \ .$$

When $n = 1$, the right side is equal to 0. When $n = 2$, it is equal to

$$4(b^2 + ba + a^2) - 3(b+a)^2 = (b-a)^2 > 0 \ .$$

When $n = 3$, we have

$$8(b^3 + b^2a + ba^2 + a^3) - 4(b+a)^3 = 4b^3 - 4b^2a - 4ba^2 + 4a^3 = 4(b^2 - a^2)(b-a) = 4(b+a)(b-a)^2 > 0 \ .$$

When $n = 4, 5$ and 6, the right side is, respectively,

$$(11b^2 + 18ab + 11a^2)(b-a)^2$$

$$(26b^3 + 54b^2a + 54ba^2 + 26a^3)(b-a)^2$$

$$(57b^4 + 136b^3a + 174b^2a^2 + 136ba^3 + 57a^4)(b-a)^2 \ .$$

There is a pattern here; can anyone express it in a general way that will yield the result, or at least show that the right side is the product of $(b-a)^2$ *and a polynomial with positive coefficients?*

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

Solution 1. Let the figure be denoted by $\mathfrak{F}$ and the circle by $\mathfrak{C}$, and let ρ be the central reflection through the centre of the circle. Suppose that m is any line of support for $\mathfrak{F}$ and that it intersects the circle in P and Q. Then there are lines p and q through P and Q respectively, perpendicular to m, which support $\mathfrak{F}$. Let p meet the circle in P and R, and q meet it in Q and S; let t be the line RS. Since $PQRS$ is concyclic with adjacent right angles, it is a rectangle, and t is a line of support of $\mathfrak{F}$. Since PS and RQ are both diameters of $\mathfrak{C}$, it follows that $S = \rho(P)$, $R = \rho(Q)$ and $t = \rho(m)$.

Hence, every line of support of $\mathfrak{F}$ is carried by ρ into a line of support of $\mathfrak{F}$. We note that $\mathfrak{F}$ must be on the same side of its line of support as the centre of the circle.

Suppose that $X \in \mathfrak{F}$. Let $Y = \rho(X)$. Suppose, if possible that $Y \notin \mathfrak{F}$. Then there must be a disc containing Y that does not intersect $\mathfrak{F}$, so we can find a line m of support for $\mathfrak{F}$ such that $\mathfrak{F}$ is on one side and Y is strictly on the other side of m. Let $n = \rho(m)$. Then n is a line of support for $\mathfrak{F}$ which has $X = \rho(Y)$ on one side and $O = \rho(O)$ on the other. But this is not possible. Hence $Y \in \mathfrak{F}$ and so $\rho(\mathfrak{F}) \subseteq \mathfrak{F}$. Now $\rho \circ \rho$ is the identity mapping, so $\mathfrak{F} = \rho(\rho(\mathfrak{F})) \subseteq \rho(\mathfrak{F})$. It follows that $\mathfrak{F} = \rho(\mathfrak{F})$ and the result follows.

Solution 2. Let P be any point on the circle $\mathfrak{C}$. There are two perpendicular lines of support from P meeting the circle in Q and S. As in the first solution, we see that P is one vertex of a rectangle $PQRS$ each of whose sides supports $\mathfrak{F}$. Let $\mathfrak{G}$ be the intersection of all the rectangles as P ranges over the circumference of the circle $\mathfrak{C}$. Since each rectangle has central symmetry about the centre of $\mathfrak{C}$, the same is true of $\mathfrak{G}$. It is clear that $\mathfrak{F} \subseteq \mathfrak{G}$. It remains to show that $\mathfrak{G} \subseteq \mathfrak{F}$. Suppose a point X in $\mathfrak{G}$ does not belong to $\mathfrak{F}$. Then there is a line r of support to $\mathfrak{F}$ for which X and $\mathfrak{F}$ are on opposite sides. This line of support intersects $\mathfrak{C}$ at the endpoints of a chord which must be a side of a supporting rectangle for $\mathfrak{F}$. The point X lies outside this rectangle, and so must lie outside of $\mathfrak{G}$. The result follows.

Solution 3. [D. Arthur] If the result is false, then there is a line through the centre of the circle such that $OP > OQ$, where P is where the line meets the boundary of the figure on one side and Q is where it meets the boundary on the other. Let m be the line of support of the figure through Q. Then, as shown in Solution 1, its reflection t in the centre of the circle is also a line of support. But then P and O lie on opposite sides of t and we obtain a contradiction.

602. Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .$$

(b) Suppose that $1 \leq r \leq n$; consider all subsets with r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) .$$

(a) *Solution 1.* $i(i-1)(i-2)\cdots(i-m+1) = \frac{[(i+1)-(i-m)]}{m+1} i(i-1)(i-2)\cdots(i-m+1)$

$$= \frac{(i+1)i(i-1)\cdots(i-m+1) - i(i-1)(i-2)\cdots(i-m+1)(i-m)}{m+1}$$

so that

$$\begin{aligned}\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) &= \sum_{i=2}^{n+1} \frac{i(i-1)\cdots(i-m)}{m+1} - \sum_{i=1}^{n} \frac{i(i-1)\cdots(i-m)}{m+1} \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} - 0 \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .\end{aligned}$$

(a) *Solution 2.* [W. Choi] Recall the identity

$$\sum_{i=m}^{n} \binom{i}{m} = \binom{n+1}{m+1}$$

which is obvious for $n = m$ and can be established by induction for $n \geq m+1$. There is an alternative combinatorial argument. Consider the number $\binom{n+1}{m+1}$ of selecting $m+1$ numbers from the set $\{1, 2, 3, \cdots, n+1\}$. The largest number must be $i+1$ where $m \leq i \leq n$, and the number of $(m+1)-$sets for which the largest number is $i+1$ is $\binom{i}{m}$. Summing over all relevant i yields the result.

We have that

$$\sum_{i=1}^{m} i(i-1)\cdots(i-m+1) = \sum_{i=m}^{n} \frac{i!}{(i-m)!} = m!\sum_{i=m}^{n}\binom{i}{m} = m!\binom{n+1}{m+1}$$
$$= \frac{(n+1)!}{(m+1)(n-m)!} = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .$$

(a) *Solution 3.* [K. Yeats] Let $n = m + k$. Then

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = m! + \frac{(m+1)!}{1!} + \cdots + \frac{n!}{(n-m)!}$$
$$= \frac{1}{(m+1)k!}\left[(m+1)!k! + \frac{(m+1)!k!(m+1)}{1!} + \frac{(m+2)!k!(m+1)}{2!} + \cdots + n!(m+1)\right]$$
$$= \frac{(m+1)!}{(m+1)k!}\left[k! + \frac{k!}{1!}(m+1) + \frac{k!}{2!}(m+2)(m+1) + \cdots + n(n-1)\cdots(m+2)(m+1)\right] .$$

The quantity in square brackets has the form (with $q = 0$)

$$\frac{k!}{q!} + \frac{k!}{(q+1)!}(m+1) + \frac{k!}{(q+2)!}(m+q+2)(m+1) + \frac{k!}{(q+3)!}(m+q+3)(m+q+2)(m+1) + \cdots$$
$$+\frac{k!}{k!}n(n-1)\cdots(m+q+2)(m+1)$$
$$= (m+q+2)\left[\frac{k!}{(q+1)!} + \frac{k!}{(q+2)!}(m+1) + \frac{k!}{(q+3)!}(m+q+3)(m+1) + \cdots + \frac{k!}{k!}n\cdots(m+q+3)(m+1)\right].$$

Applying this repeatedly with $q = 0, 1, 2, \cdots, k-1$ leads to the expression for the left sum in the problem of

$$\frac{(m+k+1)!}{(m+1)k!}\left[\frac{k!}{k!}\right] = \frac{(n+1)!}{(m+1)(n-m)!} = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .$$

[A variant, due to D. Nicholson, uses an induction on r to prove that, for $m \le r \le n$,

$$\sum_{i=m}^{r} i(i-1)\cdots(i-m+1) = \frac{(r+1)!}{(r-m)!(m+1)} .]$$

(a) *Solution 4.* For $1 \le i \le m-1$, $i(i-1)\cdots(i-m+1) = 0$. For $m \le i \le n$, $i(i-1)\cdots(i-m+1) = m!\binom{i}{m}$. Also,

$$\frac{(n+1)n\cdots(n-m+1)}{m+1} = m!\binom{n+1}{m+1}$$

so the statement is equivalent to

$$\sum_{m}^{n}\binom{i}{m} = \binom{n+1}{m+1} .$$

This is clear for $n = m$. Suppose it holds for $n = k \ge m$. Then

$$\sum_{i=m}^{k+1}\binom{i}{m} = \binom{k+1}{m+1} + \binom{k+1}{m} = \binom{k+2}{m+1}$$

and the result follows by induction.

(a) *Solution 5.* Use induction on n. If $n = 1$, then $m = 1$ and both sides of the equation are equal to 1. Suppose that the result holds for $n = k$ and $1 \leq m \leq k$. Then, for $1 \leq m \leq k$,

$$\begin{aligned}\sum_{i=1}^{k+1} i(i-1)\cdots(i-m+1) &= \frac{(k+1)k(k-1)\cdots(k-m+1)}{m+1} + (k+1)k(k-1)\cdots(k-m+2) \\ &= \frac{(k+1)k(k-1)\cdots(k-m+2)}{m+1}[(k-m+1)+(m+1)] \\ &= \frac{(k+2)(k+1)k(k-1)\cdots(k-m+2)}{m+1}\end{aligned}$$

as desired. When $m = n = k+1$, all terms on the left have $k+1$ terms and so they vanish except for the one corresponding to $i = k+1$. This one is equal to $(k+1)!$ and so to the right side.

(b) *Solution 1.* For $1 \leq i \leq r \leq n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$. Then $S(n,r,i) = \binom{n}{r}T(n,r,i)$. For $1 \leq k \leq n$, $1 \leq i \leq r$, the number of ordered $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_i = k$ is $\binom{k-1}{i-1}\binom{n-k}{r-i}$ where $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ when $b > a$. Hence

$$\binom{n}{r} = \sum_{k=1}^{n} \binom{k-1}{i-1}\binom{n-k}{r-i} \quad .$$

Replacing n by $n+1$ and r by $r+1$ yields a reading

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1} \binom{k-1}{i-1}\binom{n+1-k}{r-(i-1)} \qquad \text{for } \ 1 \leq i \leq r+1 \ .$$

Replacing $i-1$ by i yields

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1} \binom{k-1}{i}\binom{n+1-k}{r-i} \qquad \text{for } \ 0 \leq i \leq r \ .$$

When $1 \leq i \leq r$, the first term of the sum is 0, so that

$$\binom{n+1}{r+1} = \sum_{k=2}^{n+1} \binom{k-1}{i}\binom{n-(k-1)}{r-i} = \sum_{k=1}^{n} \binom{k}{i}\binom{n-k}{r-i} \quad .$$

Thus

$$S(n,r,i) = \sum_{k=1}^{n} k\binom{k-1}{i-1}\binom{n-k}{r-i} = i\sum_{k=1}^{n} \binom{k}{i}\binom{n-k}{r-i} = i\binom{n+1}{r+1}$$

so

$$T(n,r,i) = i\left(\frac{n+1}{r+1}\right) \quad .$$

(b) *Solution 2.* [Z. Liu] Define $S(n,r,i)$ for $1 \leq i \leq r \leq n$ as in Solution 1. We prove by induction that

$$S(n,r,i) = i\binom{n+1}{r+1}$$

from which

$$T(n,r,i) = i\left(\frac{n+1}{r+1}\right) \quad .$$

For each positive integer n, we have that $S(n,1,1) = 1+2+\cdots+n = \binom{n+1}{2}$ and $S(n,n,i) = i$. Suppose that $n \geq 2$, $r \geq 2$ and that $S(k,r,1) = \binom{k+1}{r+1}$ for $1 \leq k \leq n-1$. Of the $\binom{n}{r}$ $r-$tples from $\{1, 2, \cdots, n\}$, $\binom{n-1}{r-1}$ of them have smallest element equal to 1, and $\binom{n-1}{r}$ of them have smallest element exceeding 1. The latter set of $r-$tples can be put into one-one corrrespondence with $r-$tples of $\{1, 2, \cdots, n-1\}$ by subtracting one from each entry. Therefore the sum of the first (smallest) elements of the latter $r-$tples is $\binom{n-1}{r} + S(n-1, r, 1)$. Hence

$$S(n, r, 1) = \binom{n-1}{r-1} + \binom{n-1}{r} + S(n-1, r, 1) = \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} .$$

Suppose as an induction hypothesis that

$$S(m, s, j) = j\binom{m+1}{s+1}$$

for $1 \leq j \leq s \leq n-1$. This holds for $n = 2$. Let $r \geq 2$ and $1 \leq i \leq r \leq n-1$. Consider the ordered $r-$subsets of $\{1, 2, \cdots, n\}$. There are $\binom{n-1}{r-1}$ of them that begin with 1; making use of the one-one correspondence between these and $(r-1)-$ subsets of $\{1, 2 \cdots, n-1\}$ obtained by subtracting 1 from each entry beyond the first, we have that the sum of the ith elements of these is

$$\binom{n-1}{r-1} + S(n-1, r-1, i-1) = \binom{n-1}{r-1} + (i-1)\binom{n}{r} .$$

There are $\binom{n-1}{r}$ of the ordered subsets that do not begin with 1; making use of the one-one correspondence between these subsets and the $r-$subsets of $\{1, 2, \cdots, n-1\}$ obtained by subtracting 1 from each entry, we find that the sum of the ith elements is

$$\binom{n-1}{r} + S(n-1, r, i) = \binom{n-1}{r} + i\binom{n}{r+1} .$$

Hence the sum of the ith elements of all these $r-$subsets is

$$S(n, r, i) = \left[\binom{n-1}{r-1} + \binom{n-1}{r} - \binom{n}{r}\right] + i\left[\binom{n}{r} + \binom{n}{r+1}\right] = 0 + i\binom{n+1}{r+1} .$$

Putting all these elements together yields the result.

(b) *Solution 3.* When $r = 1$, we have that

$$T(n, 1, 1) = \frac{1 + 2 + \cdots + n}{n} = \frac{n+1}{2} .$$

When $r = 2$, the subsets are $\{1, 2\}, \{1, 3\}, \cdots, \{1, n\}, \{2, 3\}, \{2, 4\}, \cdots \{2, n\}, \cdots, \{n-1, n\}$, so that

$$\begin{aligned}
T(n, 2, 1) &= \frac{1 \times (n-1) + 2 \times (n-2) + \cdots + (n-1) \times 1}{\binom{n}{2}} \\
&= \frac{[(n-1) + (n-2) + \cdots + 1] + [(n-2) + (n-3) + \cdots + 1] + \cdots + 1}{\binom{n}{2}} \\
&= \frac{\sum_{j=1}^{n-1} [1 + 2 + \cdots + (n-j)]}{n(n-1)/2} = \frac{\sum_{j=1}^{n-1} (n-j+1)(n-j)/2}{n(n-1)/2} \\
&= \frac{(1/6)(n+1)n(n-1)}{(1/2)n(n-1)} = \frac{n+1}{3} ,
\end{aligned}$$

and

$$\begin{aligned}
T(n, 2, 2) &= \frac{(n-1) \times n + (n-2) \times (n-1) + \cdots + 1 \times 2}{\binom{n}{2}} \\
&= \frac{(n+1)n(n-1)/3}{n(n-1)/2} = 2\left(\frac{n+1}{3}\right) .
\end{aligned}$$

Thus, the result holds for $n = 1, 2$ and all i, r with $1 \leq i \leq r \leq n$, and for all n and $1 \leq i \leq r \leq 2$. Suppose as an induction hypothesis, we have established the result up to $n-1$ and all appropriate r and i, and for n and $1 \leq i \leq r-1$. The $r-$element subsets of $\{1, 2, \cdots, n\}$ have $\binom{n-1}{r}$ instances without n and $\binom{n-1}{r-1}$ instances with n.

Let $1 \leq i \leq r-1$. Then

$$\begin{aligned} T(n,r,i) &= \frac{\binom{n-1}{r}T(n-1,r,i) + \binom{n-1}{r-1}T(n-1,r-1,i)}{\binom{n}{r}} \\ &= \frac{i[\binom{n-1}{r}\frac{n}{r+1} + \binom{n-1}{r-1}\frac{n}{r}]}{\binom{n}{r}} = \frac{i[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} \\ &= i\frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i\left(\frac{n+1}{r+1}\right) \end{aligned}$$

Also

$$\begin{aligned} T(n,r,r) &= \frac{\binom{n-1}{r}T(n-1,r,r) + \binom{n-1}{r-1}n}{\binom{n}{r}} \\ &= \frac{\binom{n-1}{r}\frac{rn}{r+1} + \binom{n-1}{r-1}\frac{rn}{r}}{\binom{n}{r}} = \frac{r[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} = r\left(\frac{n+1}{r+1}\right) \quad . \end{aligned}$$

(b) *Solution 4.* For $1 \leq i \leq r \leq n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$. Then $S(n,r,i) = \binom{n}{r}T(n,r,i)$. We observe first that

$$S(n,r,i) = S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i)$$

for $1 \leq i \leq r-1$. This is true, since, for each j with $1 \leq j \leq n-r+1$, $S(n-j,r-1,i)$ adds the t_i over all $r-$tples for which $t_r = n-j+1$.

Now $S(n,1,1) = 1 + 2 + \cdots + n = \frac{1}{2}(n+1)n$ and $S(n,2,1) = \frac{1}{2}n(n-1) + \cdots + 1 = \frac{1}{3!}(n+1)n(n-1)$. As an induction hypothesis, suppose that $S(n,r-1,1) = \frac{1}{r!}(n+1)n(n-1)\cdots(n-r+2)$. Then

$$\begin{aligned} S(n,r,1) &= \sum_{k=r-1}^{n-1} S(k,r-1,1) \\ &= \frac{1}{r!}\sum_{k=r-1}^{n-1}(k+1)k(k-1)\cdots(k-r+2) = \frac{1}{r!}\sum_{k=1}^{n}k(k-1)\cdots(k-r+1) \\ &= \frac{1}{(r+1)!}(n+1)n(n-1)\cdots(n-r+1) = \left(\frac{n+1}{r+1}\right)\frac{n!}{r!(n-r)!} = \left(\frac{n+1}{r+1}\right)\binom{n}{r} \ . \end{aligned}$$

Thus, for each r with $1 \leq r \leq n$, $S(n,r,1) = \binom{n}{r}(n+1)/(r+1)$ so that $T(n,r,1) = (n+1)/(r+1)$.

Let $n \geq 2$. Suppose that for $1 \leq k \leq n-1$ and $1 \leq i \leq r \leq k$, it has been established that $S(k,r,i) = iS(k,r,1)$. Then for $1 \leq i \leq r \leq n$,

$$\begin{aligned} S(n,r,i) &= S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i) \\ &= i[S(n-1,r-1,1) + S(n-2,r-1,1) + \cdots + S(r-1,r-1,1) = iS(n,r,1) \ . \end{aligned}$$

Dividing by $\binom{n}{r}$ yields

$$T(n,r,i) = iT(n,r,1) = i\left(\frac{n+1}{r+1}\right) \quad .$$

Comments. (1) There is a one-one correspondence

$$(t_1, t_2, \cdots, t_r) \longleftrightarrow (n+1-t_r, n+1-t_{r-1}, \cdots, n+1-t_r)$$

of the set of suitable $r-$tples to itself, it follows that

$$\begin{aligned} S(n,r,r) &= \binom{n}{r}(n+1) - S(n,r,1) = \binom{n}{r}(n+1)\left[1 - \frac{1}{r+1}\right] \\ &= \frac{r(n+1)}{r+1}\binom{n}{r} = rS(n,r,1) \end{aligned}$$

from which $T(n,r,r) = r(n+1)/(r+1) = rT(n,r,1)$.

(2) To illustrate another method for getting and using the recursion, we prove first that $T(n,r,2) = 2T(n,r,1)$ for $2 \leq r \leq n$. Consider the case $r = 2$. For $1 \leq t_1 < t_2 \leq n$, $(t_1, t_2) \leftrightarrow (t_2 - t_1, t_2)$ defines a one-one correspondence between suitable pairs. Since $t_2 = t_1 + (t_2 - t_1)$, it follows from this correspondence that $S(n,2,2) = 2S(n,2,1)$. Dividing by $\binom{n}{2}$ yields $T(n,2,2) = 2T(n,2,1)$.

Suppose that $r \geq 2$. For each positive integer j with $1 \leq j \leq n-r+1$, we define a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$ and $t_3 - t_2 = j$ and $(r-1)-$tples $(s_1, s_2, s_3, \cdots, s_r) = (t_1, t_2, t_4 - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < s_2 = t_2 < s_3 = t_4 - j < \cdots < s_r = t_r - j \leq n - j$. The sum of the elements t_2 over all $r-$tples with $t_3 - t_2 = j$ is equal to the sum of t_2 over all the $(r-1)-$tples. Hence

$$S(n,r,2) = S(n-1, r-1, 2) + S(n-2, r-1, 2) + \cdots + S(r-1, r-1, 2) \quad .$$

More generally, for $1 \leq j \leq n-r-1$, there is a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_{i+1} - t_i = j$ and $(r-1)-$tples $(s_1, s_2, \cdots, s_{r-1}) = (t_1, \cdots, t_i, t_{i+2} - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < \cdots < s_i = t_i < s_{i+1} = t_{i+2} - j < \cdots < s_{r-1} = t_r - j \leq n - j$. We now use induction on r. We have that

$$S(n,r,i) = S(n-1, r-1, i) + S(n-2, r-1, i) + \cdots + S(r-1, r-1, i) \quad .$$

(b) *Solution 5.* [Y. Shen] We establish that

$$\sum_{k=i}^{i+(n-r)} \binom{k}{i}\binom{n-k}{r-i} = \binom{n+1}{r+1} \quad .$$

Consider the $(r+1)-$element sets where $t_{i+1} = k+1$ and $t_{r+1} \leq n+1$. We must have $i \leq k \leq n-(r-i)$ and there are $\binom{k}{i}\binom{n-k}{r-i}$ ways of selecting $t_1, \cdots, t_i$ and $t_{i+2}, \cdots, t_{r+1}$. The desired equation follows from a counting argument over all possibilities for t_{i+1}.

In a similar way, we note that $t_i = k$ for $\binom{k-1}{i-1}\binom{n-k}{r-i}$ sets $\{t_1, \cdots, t_r\}$ chosen from $\{1, \cdots, n\}$, where $1 \leq k \leq n-r+1$. Observe that

$$\binom{k-1}{i-1}\binom{n-k}{r-i} = \frac{i}{k}\binom{k}{i}\binom{n-k}{r-i} \quad .$$

Then

$$\begin{aligned} T(n,r,i) &= \frac{\sum_{k=i}^{n-r+1} k\binom{k-1}{i-1}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{i\sum_{k=i}^{n-r+i} \binom{k}{i}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i\left(\frac{n+1}{r+1}\right) \quad . \end{aligned}$$

(b) *Solution 6.* [Christopher So] Note that

$$\sum_{k=i}^{n-r+i} \binom{k}{i}\binom{n-k}{r-i}$$

is the coefficient of $x^i y^{r-i}$ in the polynomial

$$\sum_{k=i}^{n-r+i} (1+x)^k (1+y)^{n-k}$$

or in

$$\begin{aligned} \sum_{k=0}^{n+1} (1+x)^k (1+y)^{n-k} &= \frac{(1+y)^{n+1} - (1+x)^{n+1}}{y-x} \\ &= \frac{\sum_{j=0}^{n+1} \binom{n+1}{j} (y^j - x^j)}{y-x} \ . \end{aligned}$$

Now the only summand which involves terms of degree r corresponds to $j = r+1$, so that the coefficient of $x^i y^{r-1}$ in the sum is the coefficient in the single term

$$\binom{n+1}{r+1} \frac{y^{r+1} - x^{r+1}}{y-x}$$

namely, $\binom{n+1}{r+1}$. We can now complete the argument as in the fourth solution.

(b) *Comment.* Let r and n be fixed values and consider i to be variable. The $\binom{n}{r}$ $r-$term sets contain altogether $r\binom{n}{r}$ numbers, each number occurring equally often: $\frac{r}{n}\binom{n}{r}$ times. The sum of all the elements in the set is

$$S(n,r,1) + S(n,r,2) + \cdots + S(n,r,r) = \frac{r}{n}\binom{n}{r}(1+2+\cdots+n) = \frac{r(n+1)}{2}\binom{n}{r}$$

where $S(n,r,i)$ is the sum of the elements t_i over the $\binom{n}{r}$ subsets. The ordered $r-$element subsets $(t_1, t_2, \cdots, t_r)$ can be mapped one-one to themselves by

$$(t_1, t_2, \cdots, t_r) \longleftrightarrow (n+1-t_r, n+1-t_{r-1}, \cdots, n+1-t_1) \ .$$

From this, we see that, for $1 \leq r$,

$$S(n,r,r+1-i) = \binom{n}{r}(n+1) - S(n,r,i)$$

so that

$$S(n,r,1) + S(n,r,r) = S(n,r,2) + S(n,r,r-1) = \cdots = S(n,r,i) + S(n,r,r+1-i) = \cdots = \binom{n}{r}(n+1) \ .$$

This is not enough to imply that $S(n,r,i)$ is an arithmetic progression in i, but along with this fact would give a quick solution to the problem.

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1 + x$;

(b) $1 + x + x^2$;

(c) $1 + x + x^2 + x^3$;

(d) $1 + x + x^2 + x^3 + x^4$;

(e) $1 + x + x^2 + x^3 + x^4 + x^5$.

Solution. (a) $1 + x$ is a square when $x = u^2 - 1$ for some integer u (or when x is the product of two integers $u - 1$ and $u + 1$ that differ by 2).

(b) *Solution 1.* Suppose that $x^2 + x + 1 = u^2$. Then $(2x + 1)^2 + 3 = 4x^2 + 4x + 4 = 4u^2 = (2u)^2$, whence

$$3 = (2u)^2 - (2x + 1)^2 = (2u + 2x + 1)(2u - 2x - 1) \ .$$

The factors on the right must be ± 3 and ± 1 in some order, and this leads to the possibilities $(x, u) = (-1, \pm 1), (0, \pm 1)$.

(b) *Solution 2.* If $x > 0$, then $x^2 < x^2 + x + 1 < (x + 1)^2$, so that $x^2 + x + 1$ cannot be square. If $x < -1$, then $x^2 > x^2 + x + 1 > (x + 1)^2$ and $x^2 + x + 1$ cannot be square. This leaves only the possibilities $x = 0, -1$.

(b) *Solution 3.* For given u, consider the quadratic equation

$$x^2 + x + 1 = u^2 \ .$$

Its discriminant is $1 - 4(1 - u^2) = 4u^2 - 3$. It will have integer solutions only if $4u^2 - 3 = v^2$ for some integer v, *i.e.*, $(v + 2u)(v - 2u) = -3$. The only possibilities are $(u, v) = (\pm 1, \pm 1), (\pm 1, \mp 1)$.

(b) *Solution 4.* [J. Chui] If $f(x) = 1 + x + x^2$, then $f(x) = f(-(x + 1))$, so we need deal only with nonnegative values of x. We have that $f(0) = f(-1) = 1$ is square. Let $x \geq 1$ and suppose that $1 + x + x^2 = u^2$ for some integer u. Then $(1 + x)^2 - u^2 = x > 0$ so that $1 + x > u$. This implies that $x \geq u$, whence $x^2 \geq u^2 = x^2 + x + 1$, a contradiction. Thus the only possibilities are $x = 0, -1$.

(b) *Solution 5.* [A. Birka] Suppose that $x^2 + x + 1 = u^2$ with $u \geq 0$. This is equivalent to $x = (1 + x)^2 - u^2 = (1 + x + u)(1 + x - u)$, so that $1 + x + u$ and $1 + x - u$ both divide x. If $x \geq 1$, then $1 + x + u$ exceeds x and so cannot divide x. If $x \leq 0$, then $(-x) + u - 1$ divides x, which is impossible unless $u = 1$ or $u = 0$. Only $u = 1$ is viable, and this leads to $x = 0, -1$.

(c) *Solution.* $1 + x + x^2 + x^3 = (1 + x^2)(1 + x)$. Let d be a common prime divisor of $1 + x$ and $1 + x^2$. Then d must also divide $x(x - 1) = (1 + x^2) - (1 + x)$. Since $\gcd(x, x + 1) = 1$, d must divide $x - 1$ and so divide $2 = (x + 1) - (x - 1)$. Hence, the only common prime divisor of $1 + x^2$ and $1 + x$ is 2.

Suppose $1 + x + x^2 + x^3 = (1 + x^2)(1 + x)$ is square. Then there are only two possibilities:

$$\text{(i)} \quad 1 + x^2 = u^2 \quad \text{and} \quad 1 + x = v^2 \quad \text{for integers } u \text{ and } v \ ;$$

$$\text{(ii)} \quad 1 + x^2 = 2r^2 \quad \text{and} \quad 1 + x = 2s^2 \quad \text{for integers } r \text{ and } s \ .$$

Ad (i): $1 = u^2 - x^2 = (u - x)(u + x) \Leftrightarrow (x, u) = (0, \pm 1)$.
Ad (ii): We have $x^2 - 2r^2 = -1$ which has solutions

$$(x, r) = (-1, 1), (1, 1), (7, 5), (41, 29), \cdots \ .$$

The complete set of solutions of $x^2 - 2r^2 = \pm 1$ in positive integers is given by $\{(x_n, r_n) : n = 1, 2, \cdots\}$, where $x_n + r_n\sqrt{2} = (1 + \sqrt{2})^n$, with odd values of n yielding solutions of $x^2 - 2r^2 = -1$. We need to select values of x for which $x + 1 = 2s^2$ for some s. $x = -1, 1, 7$ work, yielding

$$1 - 1 + (-1)^2 + (-1)^3 = 0$$

$$1 + 1 + 1^2 + 1^3 = 2^2$$

$$1 + 7 + 7^2 + 7^3 = 8 \times 50 = 20^2 \ .$$

There may be other solutions.

(d) *Solution 1.* Let $f(x) = x^4 + x^3 + x^2 + x + 1 = (x^5 - 1)/(x - 1)$, with the quotient form for $x \neq 1$. We have that $f(0) = f(-1) = 1^2$ and $f(3) = (243 - 1)/2 = 11^2$. Also $f(1) = 5$ and $f(2) = 31$. Suppose that $x \geq 4$. Then $x(x - 2) > 3$, so that $x^2 > 2x + 3$. Hence

$$\begin{aligned}(2x^2 + x + 1)^2 &= 4x^4 + 4x^3 + 5x^2 + 2x + 1 \\ &> 4x^4 + 4x^3 + 4x^2 + 4x + 4 = 4f(x)\end{aligned}$$

and

$$\begin{aligned}4f(x) &= (4x^4 + 4x^3 + x^2) + (3x^2 + 4x + 4) \\ &= (2x^2 + x)^2 + (3x^2 + 4x + 4) > (2x^2 + x)^2 \ .\end{aligned}$$

Thus, $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$ and so cannot be square. Hence $f(x)$ cannot be square.

Similarly, if $x \leq -2$, then $x(x - 2) > 3$ and $3x^2 + 4x + 4 > 0$, and we again find that $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$. Hence $f(x)$ is square if and only if $x = -1, 0, 3$.

(d) *Solution 2.* [M. Boase] For $x > 3$,

$$\left(x^2 + \frac{x}{2}\right)^2 < x^4 + x^3 + x^2 + x + 1 < \left(x^2 + \frac{x + 1}{2}\right)^2$$

so that, lying between two half integers, $x^4 + x^3 + x^2 + x + 1$ is not square. Suppose $x = -y$ is less than -1. Since $y - 1 < \frac{3}{4}y^2$ and $y^2 + 2y - 3 = (y + 3)(y - 1) > 0$,

$$\left(y^2 - \frac{y}{2}\right)^2 < 1 - y + y^2 - y^3 + y^4 < \left(y^2 - \frac{y - 1}{2}\right)^2$$

so again the middle term is not square. The cases $x = -1, 0, 1, 2, 3$ can be checked directly.

(e) *Solution 1.* Let

$$\begin{aligned}g(x) &= x^5 + x^4 + x^3 + x^2 + x + 1 = (x + 1)(x^4 + x^2 + 1) \\ &= (x + 1)[(x^2 + 1)^2 - x^2] = (x + 1)(x^2 + x + 1)(x^2 - x + 1) \ .\end{aligned}$$

Observe that $g(x) < 0$ for $x \leq -2$, so $g(x)$ cannot be square in this case. Let us analyze common divisors of the three factors of $g(x)$.

Suppose that p is a prime divisor of $x + 1$. Then

$$x^2 + x + 1 = x(x + 1) + 1 \equiv 1 \mod p$$

and

$$x^2 - x + 1 = x(x + 1) - 2(x + 1) + 3 \equiv 3 \mod p \ .$$

Hence $\gcd(x + 1, x^2 + x + 1) = 1$ and $\gcd(x + 1, x^2 - x + 1)$ is either 1 or 3.

Suppose q is prime and $x^2 + x + 1 \equiv 0 \pmod q$. Then $x(x + 1) \equiv -1 \pmod q$, and $x^2 - x + 1 \equiv -2x \pmod q$. Since $x^2 + x + 1$ is odd, $q \neq 2$, then $x^2 - x + 1 \not\equiv 0 \pmod q$. Hence $\gcd(x^2 + x + 1, x^2 - x + 1) = 1$.

As we have seen from (b), $x^2 + x + 1$ is square if and only if $x = -1$ or 0. Indeed $g(-1) = 0^2$ and $g(0) = 1^2$. Otherwise, $x^2 + x + 1$ cannot be square. But $\gcd(x^2 + x + 1, (x + 1)(x^2 - x + 1)) = 1$, so $g(x)$ cannot be a square either. Hence $x^5 + x^4 + x^3 + x^2 + x + 1$ is square if and only if $x = -1$ or 0.

(e) *Solution 2.* [M. Boase] Observe that $x^5 + x^4 + \cdots + 1 = (x^3 + 1)(x^2 + x + 1)$. Since $x^3 + 1 = (x^2 + x + 1)(x - 1) + 2$, the greatest common divisor of $x^3 + 1$ and $x^2 + x + 1$ must divide 2. But $x^2 + x + 1 = x(x + 1) + 1$

is always odd, so the greatest common divisor must be 1. Hence x^2+x+1 and $x+1$ must both be square. Hence x must be either -1 or 0.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

Solution 1. Let Γ be the circle of equation $x^2+y^2=1$ and let l be the line of equation $y=-1$. The points of the square must lie on the circle of equation $x^2+y^2=2$. Let them be

$$A \sim (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$$

$$B \sim (-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta)$$
$$C \sim (-\sqrt{2}\cos\theta, -\sqrt{2}\sin\theta)$$
$$D \sim (\sqrt{2}\sin\theta, -\sqrt{2}\cos\theta)$$

for some angle θ with $-\pi/4 \le \theta \le \pi/4$. Observe that $1/\sqrt{2} \le \cos\theta \le 1$ and that $-1/\sqrt{2} \le \sin\theta \le 1/\sqrt{2}$.

Then $A' \sim (\sqrt{2}\cos\theta, -1)$, $B' \sim (-\sqrt{2}\sin\theta, -1)$, $C' \sim (-\sqrt{2}\cos\theta, -1)$ and $D' \sim (\sqrt{2}\sin\theta, -1)$, so that $AA' = 1+\sqrt{2}\sin\theta$, $BB' = 1+\sqrt{2}\cos\theta$, $CC' = 1-\sqrt{2}\sin\theta$ and $DD' = 1-\sqrt{2}\cos\theta$. Hence

$$\begin{aligned} AA' \cdot CC' - BB' \cdot DD' &= (1+\sqrt{2}\sin\theta)(1-\sqrt{2}\sin\theta) - (1+\sqrt{2}\cos\theta)|1-\sqrt{2}\cos\theta| \\ &= (1+\sqrt{2}\sin\theta)(1-\sqrt{2}\sin\theta) + (1+\sqrt{2}\cos\theta)(1-\sqrt{2}\cos\theta) \\ &= 1-2\sin^2\theta + 1 - 2\cos^2\theta = 0 \quad . \end{aligned}$$

Solution 2. One can proceed as in the first solution, taking the four points on the larger circle at the intersection with the perpendicular lines $y = mx$ and $y = -x/m$. The points are

$$A \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad B \sim \left(\frac{-m\sqrt{2}}{\sqrt{m^2+1}}, \frac{\sqrt{2}}{\sqrt{m^2+1}}\right)$$

$$C \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{-m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad D \sim \left(\frac{m\sqrt{2}}{\sqrt{m^2+1}}, \frac{-\sqrt{2}}{\sqrt{m^2+1}}\right) \quad .$$

In this case, the products turn out to be equal to $|(m^2-1)/(m^2+1)|$.

Solution 3. [A. Birka] Let the circle have equation $x^2+y^2=1$ and the square have vertices $A \sim (1,1)$, $B \sim (-1,1)$, $C \sim (-1,-1)$, $D \sim (1,-1)$. Suppose, wolog, that the line l is tangent to the circle at $P(t, \sqrt{1-t^2})$ with $0 < t < 1$ and intersects CB produced in Y and AD in X. The line l has equation $tx + \sqrt{1-t^2}y = 1$ and so the coordinates of X are $(1,u)$ and of Y are $(-1, 1/u)$ where $u = (1-t)/\sqrt{1-t^2}$. Now $YB : YC = (1-u) : (1+u) = AX : XD$. Since $\Delta YBB'$ is similar to $\Delta YCC'$ and $\Delta XAA'$ is similar to $\Delta XDD'$.

$$BB' : CC' = YB : YC = AX : XD = AA' : DD' \ ,$$

and the result follows.

Comment. If the circle has equation $x^2+y^2=r^2$, the square has vertices $(\pm r, \pm r)$ and the line through a point (a,b) on the circle has equation $ax+by=r^2$, then the distance product is $2ab$.