Solutions for August

633. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

Solution 1. [A. Murali] Let $\angle ADB = \theta$, $|AB| = c$, $|CA| = b$, $|AD| = d$, $|CD| = x$, $|BD| = y$. Assume that $\angle ABD = 2\angle ADB$. By the Law of Sines applied to triangle ABD,

$$\frac{d}{\sin 2\theta} = \frac{c}{\sin\theta} \Longrightarrow d = 2c\cos\theta \ .$$

By the Law of Cosines in triangle ABD,

$$4c^2\cos^2\theta = d^2 = c^2 + y^2 - 2cy\cos 2\theta \ ,$$

from which

$$\begin{aligned} 0 &= y^2 - (2c\cos 2\theta)y + c^2(1 - 4\cos^2\theta) \\ &= y^2 - (2c\cos 2\theta)y - c^2(2\cos 2\theta + 1) \\ &= [y + c][y - c(2\cos 2\theta + 1)] \ . \end{aligned}$$

Hence $y = (2\cos 2\theta + 1)c$.

By the Law of Cosines in triangle ACD,

$$b^2 = d^2 + x^2 + 2xd\cos\theta \Longrightarrow 0 = 4[x^2 + (2d\cos\theta)x + (d^2 - b^2)] \ .$$

Since $x + y = 2(b - c)$, then

$$2b = x + y + 2c = x + (2\cos 2\theta + 3)c \ .$$

Now $2d\cos\theta = 4c\cos^2\theta = 2c\cos 2\theta + 2c$ and

$$4d^2 - 4b^2 = 16c^2\cos^2\theta - x^2 - 2c(2\cos 2\theta + 3)x - (2\cos 2\theta + 3)^2c^2 \ ,$$

whence

$$\begin{aligned} 0 &= 4x^2 + (8\cos 2\theta + 8)cx + 16c^2\cos^2\theta - x^2 - (4\cos 2\theta + 6)cx - (4\cos^2 2\theta + 12\cos 2\theta + 9)c^2 \\ &= 3x^2 + (4\cos 2\theta + 2)cx + [(8\cos 2\theta + 8) - (4\cos^2 2\theta + 12\cos 2\theta + 9)]c^2 \\ &= 3x^2 + (4\cos 2\theta + 2)cx - [4\cos^2 2\theta + 4\cos 2\theta + 1]c^2 \\ &= 3x^2 + (4\cos 2\theta + 2)cx - (2\cos 2\theta + 1)^2c^2 \\ &= [3x - (2\cos 2\theta + 1)c][x + (2\cos 2\theta + 1)c] = [3x - y][x + y] = a(3x - y) \ . \end{aligned}$$

Hence $y = 3x$.

For the converse, let $y = 3x$, $\angle ADB = \theta$ and $\angle ABD = \beta$. By hypothesis, $|BC| = 4x = 2(b - c)$. By the Law of Cosines on triangle ABC, $b^2 = c^2 + 16x^2 - 8cx\cos\beta$, so that

$$\begin{aligned} \cos\beta &= \frac{16x^2 + c^2 - b^2}{8cx} = \frac{4(b - c)^2 + (c^2 - b^2)}{4c(b - c)} \\ &= \frac{4(b - c) - (c + b)}{4c} = \frac{3b - 5c}{4c} \ . \end{aligned}$$

By Stewart's Theorem, $b^2(3x) + c^2(x) = 4x[d^2 + (3x)x]$, so that

$$\begin{aligned} d^2 &= \frac{3b^2 + c^2 - 12x^2}{4} = \frac{3b^2 + c^2 - 3(b - c)^2}{4} \\ &= \frac{6bc - 2c^2}{4} = \frac{(3b - c)c}{2} \ . \end{aligned}$$

From triangle ABD, we have that $c^2 = d^2 + 9x^2 - 6dx\cos\theta$, so that

$$\begin{aligned}\cos\theta &= \frac{9x^2+d^2-c^2}{6dx} = \frac{(3x-c)(3x+c)+d^2}{6dx} \\ &= \frac{(6x-2c)(6x+2c)+4d^2}{24dx} = \frac{(3b-5c)(3b-c)+2(3b-c)c}{12d(b-c)} \\ &= \frac{(3b-c)(3b-3c)}{12d(b-c)} = \frac{3b-c}{4d} \ .\end{aligned}$$

Therefore,

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 = \frac{2(3b-c)^2}{16d^2} - 1 \\ &= \frac{2(3b-c)^2 - 8(3b-c)c}{8(3b-c)c} = \frac{2(3b-c)-8c}{8c} = \frac{3b-5c}{4c} = \cos\beta \ .\end{aligned}$$

Thus, either $2\theta = \beta$ or $2\theta = 2\pi - \beta$. But the latter case is excluded, since it would imply that β and θ are two angles of a triangle for which $\beta + \theta = 2\pi - \theta = \pi + \beta/2 > \pi$.

Solution 2. Case (i): Suppose that $\angle B$ is acute. Let $AH \perp BC$ and E lie on CH such that $AE = AB$.

$AC^2 - CH^2 = AB^2 - BH^2$ implies that

$$AC^2 - AB^2 = CH^2 - BH^2 = (CH - BH)(CH + BH) = (CH - HE)BC = CE \cdot BC = CE[2(AC - AB)] \ .$$

Hence $AC + AB = 2CE$. Also $AC - AB = \frac{1}{2}BC$. Therefore $2AB + \frac{1}{2}BC = 2CE$.

Suppose that $\angle ABD = 2\angle ADB$. Then $\angle AEB = 2\angle ADB \Rightarrow \Delta ADE$ is isosceles. Hence

$$AB = AE = DE \Rightarrow 2DE + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \ .$$

Conversely, suppose that $BD = 3CD$. Then

$$BC = 4CD \Rightarrow \frac{1}{4}BC = CE - DE \ .$$

From the above,

$$\begin{aligned}AB = CE - \frac{1}{4}BC = DE \Rightarrow AE = DE \\ \Rightarrow \angle ABD = \angle AEB = 2\angle ADB \ .\end{aligned}$$

Case (ii): Suppose $\angle B = 90°$. Then

$$AC^2 - AB^2 = BC^2 = 2(AC - AB)\cdot BC \Rightarrow AC + AB = 2BC$$

$$\Rightarrow \frac{1}{2}BC + AB + AB = 2BC \Rightarrow AB = \frac{3}{4}BC \ .$$

$$\angle ABD = 2\angle ADB \Rightarrow \angle ADB = 45° = \angle BAD \Rightarrow AB = BD$$

$$\Rightarrow BD = \frac{3}{4}BC \Rightarrow BD = 3CD \ .$$

$$BD = 3CD \Rightarrow BD = \frac{3}{4}BC = AB \Rightarrow \angle ADB = \angle BAD = 45° = \frac{1}{2}\angle ABD \ .$$

Case (iii): Suppose $\angle B$ exceeds $90°$. Let $AH \perp BC$ and E be on CH produced such that $AE = AB$. Then

$$AC^2 - CH^2 = AB^2 - BH^2 \Rightarrow (AC - AB)(AC + AB) = CH^2 - BH^2 = (CH - BH)(CH + BH) = CB \cdot CE$$

$$\Rightarrow AC + AB = 2CE \ .$$

Also

$$AC - AB = \frac{1}{2}BC \Rightarrow 2AB + \frac{1}{2}BC = 2CE \Rightarrow AB + \frac{1}{4}BC = CE \ .$$

Let $\angle ABD = 2\angle ADB$. Then

$$180^\circ - \angle ABE = 2\angle ADB \Rightarrow \angle AEB + 2\angle ADE = \angle ABE + 2\angle ADB = 180^\circ \ .$$

Also

$$\angle AEB + \angle EAD + \angle ADE = 180^\circ \Rightarrow \angle EAD = \angle ADE \Rightarrow AE = ED \ .$$

Hence

$$AB = ED \Rightarrow 2ED + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \ .$$

Conversely, suppose that $BD = 3CD$. Then $BC = 4CD$ and $ED = CE - CD = CE - \frac{1}{4}BC = AB$ so that $ED = AE$ and $\angle EAD = \angle ADE$. Therefore

$$\angle ABD = 180^\circ - \angle AED = \angle EAD + \angle ADE = 2\angle ADE = 2\angle ADB \ .$$

Solution 3. [R. Hoshino] Let $\angle ABD = 2\theta$. By the Law of Cosines, with the usual conventions for a, b, c,

$$\begin{aligned}1 - 2\sin^2\theta = \cos 2\theta &= \frac{c^2 + 4(b-c)^2 - b^2}{4c(b-c)} \\ &= \frac{b-c}{c} - \frac{b+c}{4c} = \frac{3b-5c}{4c} \qquad (\text{since } \ b \neq c) \\ &\Rightarrow 3(b-c) = 6c - 8c\sin^2\theta \\ &\Rightarrow \frac{3(b-c)}{2}\sin\theta = c(3\sin\theta - 4\sin^3\theta) = c\sin 3\theta \\ &\Rightarrow \frac{\sin\theta}{c} = \frac{2\sin 3\theta}{3(b-c)} \quad . \ (*)\end{aligned}$$

Suppose now that D is selected so that $\angle ADB = \theta$. Then, by the Law of Sines,

$$\frac{\sin\theta}{c} = \frac{\sin(180^\circ - 3\theta)}{x} = \frac{\sin 3\theta}{x}$$

where $x = |BD|$. Comparison with (*) yields $x = \frac{1}{2}(3(b-c))$ so $4BD = 3BC \Rightarrow BD = 3CD$ as desired.

On the other hand, suppose D is selected so that $BD = 3CD$. Then $BD = \frac{3}{2}(b-c)$. Let $\angle ADB = \phi$. Then

$$\frac{\sin\phi}{c} = \frac{\sin(180^\circ - \phi - 2\theta)}{\frac{3}{2}(b-c)} = \frac{\sin(\phi + 2\theta)}{\frac{3}{2}(b-c)} \ .$$

Hence

$$\begin{aligned}\frac{\sin(\phi+2\theta)}{\sin\phi} = \frac{\sin 3\theta}{\sin\theta} &\Rightarrow \sin\theta\sin(\phi+2\theta) = \sin 3\theta\sin\phi \\ &\Rightarrow \frac{1}{2}[\cos(\theta+\phi) - \cos(3\theta+\phi)] = \frac{1}{2}[\cos(3\theta-\phi) - \cos(3\theta+\phi)] \\ &\Rightarrow \cos(\theta+\phi) = \cos(3\theta-\phi) \\ &\Rightarrow \theta + \phi = \pm(3\theta - \phi) \qquad \text{or} \qquad \theta + \phi + 3\theta - \phi = 360^\circ \ .\end{aligned}$$

The only viable possibility is $\theta + \phi = 3\theta - \phi \Rightarrow \theta = \phi$ as desired.

Solution 4. [J. Chui] First, recall *Stewart's Theorem.* Let XYZ be a triangle with sides x, y, z respectively opposite XYZ. Let W be a point on YZ so that $|XW| = u$, $|YW| = v$ and $|ZW| = w$. Then $x(u^2 + vw) = vy^2 + wz^2$. This is an immediate consequence of the Law of Cosines. Let $\theta = \angle YWX$. Then $z^2 = u^2 + v^2 - 2uv\cos\theta$ and $y^2 = u^2 + w^2 + 2uw\cos\theta$. Multiply these equations by u and v respectively, add and use $x = v + w$ to obtain the result.

Now to the problem. Suppose $BD = 3CD$. Let $|AC| = 2b$, $|AB| = 2c$, so that $|BC| = 4(b - c)$, $|BD| = 3(b-c)$ and $|CD| = b-c$. If $|AD| = d$, then an application of Stewart's Theorem yields $d^2 = 2c(3b-c)$. Applying the Law of Cosines to ΔABC and ΔABD respectively yields

$$\cos\angle ABC = \frac{3b - 5c}{4c} \quad \text{and} \quad \cos\angle ADB = \frac{3b - c}{2\sqrt{2c(3b - c)}} \quad .$$

Then $\cos 2\angle ADB = (3b-5c)/4c$. Hence, either $2\angle ADB = \angle ABC$ or $\angle ABC + 2\angle ADB = 360°$. In the latter case, $\angle ABC + \angle ADB = 360° - \angle ADB > 180°$, which is false. Hence $\angle ABC = 2\angle ADB$.

On the other hand, let $2\angle ADB = \angle ABC$. If D' is a point on BC with $BD' = 3CD'$, the $2\angle AD'B = \angle ABC = 2\angle ADB$, so that $D = D'$. The result follows.

Solution 5. Let $|AB| = a$, $|AC| = a + 2$, $|BD| = 3$, $|CD| = 1$, $\angle ABD = 2\theta$, $\angle ADB = \phi$. Then $(a + 2)^2 = a^2 + 16 - 8a\cos 2\theta$, whence $a = 3(1 + 2\cos 2\theta)^{-1}$ (so $0 < \theta < 60°$). By the Law of Sines,

$$\frac{\sin(2\theta + \phi)}{3} = \frac{(1 + 2\cos 2\theta)\sin\phi}{3}$$

so that

$$\begin{aligned} 0 &= \sin\phi + 2\sin\phi\cos 2\theta - \sin(2\theta + \phi) \\ &= \sin\phi + \sin\phi\cos 2\theta - \sin 2\theta\cos\phi \\ &= \sin\phi + \sin(\phi - 2\theta) = 2\sin(\phi - \theta)\cos\theta \quad . \end{aligned}$$

Since $0 \leq |\phi - \theta| < 180°$, we find that $\phi = \theta$ as desired. The converse can be obtained as in the third solution.

Solution 6. [A. Birka] First, note that, when $BD = 3CD$, we must have $\angle ADB < 90°$, since $AC > AB$ and D is on the same side of the altitude from A as C. Also, when $\angle ABD = 2\angle ADB$, $\angle ADB < 90°$. Thus, we can assume that $\angle ADB$ is acute throughout.

We can select positive numbers u, v and w so that $|BC| = v + w$, $|AC| = u + w$ and $|AB| = u + v$. By hypothesis, $v + w = 2(w - v)$, so that $w = 3v$.

Suppose that $BD = 3CD$. Then $BC = 4CD$, whence $|CD| = v$. Hence $|BD| = 3v$. By the Law of Cosines,

$$(u + 3v)^2 = (u + v)^2 + (4v)^2 - 8v(u + v)\cos B$$

so that

$$\cos B = \frac{8v^2 - 4uv}{8v(u + v)} = \frac{2v - u}{2(u + v)} \ .$$

Hence

$$|AD|^2 = (u + v)^2 + (3v)^2 - 6v(u + v)\cos B = u^2 + 5uv + 4v^2 = (u + 4v)(u + v) \quad .$$

Since $\sin^2\angle ABD = 1 - \cos^2 B = [3u(u + 4v)]/[4(u + v)^2]$, and, by the Law of Sines,

$$\frac{\sin^2\angle ADB}{\sin^2\angle ABD} = \frac{u + v}{u + 4v} \ ,$$

we have that

$$\sin^2\angle ADB = \frac{3u}{4(u + v)} \quad \text{and} \quad \cos^2\angle ADB = \frac{u + 4v}{4(u + v)} \quad .$$

Thus $\sin^2 \angle ABD = 4 \sin^2 \angle ADB \cos^2 \angle ADB$ so that either $\angle ABD = 2\angle ADB$ or $\angle ABD + 2\angle ADB = 180°$. The latter case would yield $\angle ADB = \angle BAD$, so that $AB = BD$. This would make ΔABC a $3-4-5$ right triangle and ΔABD an isosceles right triangle, whence $90° = \angle ABD = 2\angle ADB$. The converse can be shown as in the previous solutions. The result follows.

634. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

Preliminary comments. With the surds in the second equation, we must restrict ourselves to nonnegative values of x. Because of the complexity of the expressions, it is probably impossible to eliminate one of the variables and solve for the other. Let us make a few preliminary observations:

(i) $(x, y) = (1, 1)$ is an obvious solution;

(ii) Both equations are symmetric in x and y;

(iii) Taking $f(x, y) = 2^{x^2+y} + 2^{x+y^2}$ and $g(x, y) = \sqrt{x} + \sqrt{y}$, we have that $f(0, y) = 2^y + 2^{y^2}$ and $g(0, y) = \sqrt{y}$; thus, $f(0, y) = 8 \Rightarrow 1 < y < 2$ and $g(0, y) = 2 \Leftrightarrow y = 4$. The graphs of $f(x, y) = 8$ and $g(x, y) = 2$ should be sketched.

This suggests that $f(x, y) = 8 \Rightarrow x + y \leq 2$ and $g(x, y) = 2 \Rightarrow x + y \geq 2$ with equality for both $\Leftrightarrow (x, y) = (1, 1)$. Hence we look for a relationship among $f(x, y)$, $g(x, y)$ and $x + y$.

Solution 1.

$$(\sqrt{x} + \sqrt{y})^2 = x + 2\sqrt{xy} + y \leq x + (x + y) + y = 2(x + y)$$

by the Arithmetic-Geometric Means Inequality. Hence

$$\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)} \ .$$

Also, by the same AGM inequality,

$$2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y+x+y^2}} \quad .$$

Now, using the inequality again, we find that

$$x^2 + y + x + y^2 = (x^2 + y^2) + (x + y) \geq \frac{1}{2}(x + y)^2 + (x + y)$$

so that

$$2^{x^2+y} + 2^{x+y^2} \geq 2^{1+\frac{1}{4}(x+y)^2+\frac{1}{2}(x+y)} = 2^{\frac{1}{4}[(x+y+1)^2+3]} \ .$$

Suppose the (x, y) satisfies the system. Then

$$\sqrt{2(x + y)} \geq 2 \Rightarrow (x + y) \geq 2$$

and

$$\frac{1}{4}[(x + y + 1)^2 + 3] \leq 3 \Rightarrow (x + y + 1)^2 \leq 9 \Rightarrow x + y + 1 \leq 3 \Rightarrow x + y \leq 2 \quad .$$

Hence $x + y = 2$ and all inequalities are equalities. Therefore $x = y = 1$.

Solution 2. [A. Rodriguez] Wolog, we may assume that $x \geq 1$. Let $\sqrt{x} + \sqrt{y} = 2$; then $y = (2 - \sqrt{x})^2$. Define

$$g(x) = x + y^2 + y + x^2 = (2 - \sqrt{x})^4 + x^2 + x + (2 - \sqrt{x})^2$$
$$= 2x^2 - 8x^{\frac{3}{2}} + 26x - 36x^{\frac{1}{2}} + 20 \ .$$

Then

$$g'(x) = 4x - 12x^{\frac{1}{2}} + 26 - 18x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}(2x^{\frac{3}{2}} - 6x + 13x^{\frac{1}{2}} - 9)$$
$$= 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)(2x - 4x^{\frac{1}{2}} + 9) = 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)[2(x^{\frac{1}{2}} - 1)^2 + 7] > 0$$

for $x > 1$. Hence $g(x)$ is strictly increasing for $x > 1$, so that $g(x) \geq g(1) = 4$ for $x \geq 1$ with equality if and only if $x = 1$. Thus, if the first equation holds, then

$$8 = 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{g(x)}} \Rightarrow 16 \geq 2^{g(x)} \Rightarrow g(x) \leq 4 \ \ .$$

Hence $g(x) = 4$, so that $x = 1$ and $y = 1$. Thus, $(x, y) = (1, 1)$ is the only solution.

Solution 3. [S. Yazdani] Set $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x^2 + y = (1 + u)^4 + (1 - u)^2$ and $x + y^2 = (1 - u)^4 + (1 + u)^2$, so

$$8 = 2^{x^2+y} + 2^{x+y^2} = 2^{u^4+7u^2+2}\left(2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}}\right) \geq 2^2(2) = 8$$

with equality if and only if $u = 0$. Since the extremes of this inequality are equal, we must have $u = 0$, so $x = y = 1$.

Solution 4. [C. Hsia] With $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$, we can write the first equation as

$$2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}} = 2^{1-7u^2-u^4} \ .$$

Let $z = 2^{4u^3+2u}$. We note that the quadratic $z^2 - 2^{1-7u^2-u^4}z + 1 = 0$ is solvable, and so has nonnegative discriminant. Hence

$$2^{2-14u^2-2u^4} \geq 4 = 2^2 \Rightarrow -14u^2 - 2u^4 \geq 0 \Rightarrow u = 0 \ \ .$$

Hence $x = y = 1$.

Solution 5. [M. Boase] $2(x+y) \geq (x+y)+2\sqrt{xy} = (\sqrt{x}+\sqrt{y})^2 = 4$ so that $x+y \geq 2$. Let $f(t) = t(t+1)$. For positive values of t, $f(t)$ is an increasing strictly convex function of t. Hence

$$f(x) + f(y) \geq 2f(\frac{1}{2}(x + y)) \geq 2f(1) = 4$$

so that $x^2 + x + y^2 + y \geq 4$. Equality occurs if and only if $x = y = 1$. Applying the Arithmetic-Geometric Means Inequality, we find that

$$4 = \frac{1}{2}(2^{x^2+y} + 2^{x+y^2}) \geq 2^{\frac{1}{2}(x^2+y^2+x+y)}$$

so that $x^2 + x + y^2 + y \leq 4$. Hence $x^2 + x + y^2 + y = 4$ and so $x = y = 1$.

Comment. Note that $2(x^2 + y^2) \leq (x + y)^2$ with equality if and only if $x = y$. Hence

$$x^2 + y^2 + x + y \geq \frac{1}{2}(x + y)^2 + (x + y) \geq 4$$

with equality if and only if $x = y = 1$. This avoids the use of the convexity of the function f.

Solution 6. [J. Chui] Wolog, let $x \geq y$ so that $\sqrt{x} \geq 1 \geq \sqrt{y}$. Suppose that $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x + y = 2 + 2u^2 \geq 2$ and $xy = (1 - u^2)^2 \leq 1$. Thus

$$8 = 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y+x+y^2}}$$
$$= 2\sqrt{2^{(x+y)(x+y+1)-2xy}} \geq 2\sqrt{2^{2\cdot 3-2\cdot 1}} = 2^3 = 8$$

with equality if and only if $x = y$.

Solution 7. [C. Deng] By the Root-Mean-Square, Arithmetic Mean Inequality, we have that

$$\frac{x^2+y^2}{2} \geq \left(\frac{x+y}{2}\right)^2 \geq \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^4 = 1 \ ,$$

with equality if and only if $x = y = 1$. By the Arithmetic-Geometric Means Inequality, we have

$$\begin{aligned} 4 &= \frac{2^{x^2+y} + 2^{x+y^2}}{2} \geq \sqrt{2^{x^2+y^2+x+y}} \\ &\geq \sqrt{2^{2+2}} = 4 \ . \end{aligned}$$

Since equality must hold throughtout, $x = y$, and thus the only solution to the system is $(x, y) = (1, 1)$.

635. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

Solution. Let P and Q be the centres of the spheres of respective radii r and R, and let O be the apex of the cone. Consider a vertical slice of the configuration through its axis of rotation. Let A and B be points in the slice that are the tangent points of the smaller and larger spheres, respectively, with the tangent cone. Let u and V be the centres of the circles through A and B, respectively, that are perpendicular ot the axis of rotation.

From a consideration of similar triangles and pythagoras theorem, we find that

$$\begin{aligned} |OP| &= r\left(\tfrac{R+r}{R-r}\right) & |OU| &= \tfrac{4Rr^2}{R^2-r^2} \\ |UP| &= r\left(\tfrac{R-r}{R+r}\right) & |AU| &= \tfrac{2r}{R+r}\sqrt{Rr} \\ |OQ| &= R\left(\tfrac{R+r}{R-r}\right) & |OV| &= \tfrac{4R^2r}{R^2-r^2} \\ |VQ| &= R\left(\tfrac{R-r}{R+r}\right) & |BV| &= \tfrac{2R}{R+r}\sqrt{Rr} \end{aligned}$$

The volume of the cone obtained by rotating OBV is

$$\frac{1}{3}\pi|BV|^2|OV| = \frac{16\pi R^5 r^2}{3(R+r)^3(R-r)}$$

and the volume of the cone obtained by rotating OAU is

$$\frac{16\pi R^2 r^5}{3(R+r)^3(R-r)}$$

so that the volume of the frustum obtained by rotating $AUVB$ is

$$\frac{16\pi R^2 r^2 (R^3 - r^3)}{3(R+r)^3(R-r)} = \frac{16\pi R^2 r^2}{3(R+r)^3}(R^2 + Rr + r^2) \ .$$

The volume of a slice of a sphere of radius a and height h from the equatorial plane is

$$\pi \int_0^h (a^2 - t^2)dt = \pi[a^2 h - h^3/3] \ .$$

The portion of the larger sphere included within the frustum has volume

$$\begin{aligned}\frac{2\pi R^3}{3} - \pi\left[R^3\left(\frac{R-r}{R+r}\right) - \frac{R^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] \\ = \frac{\pi R^3}{3}\left[2 - 3\left(\frac{R-r}{R+r}\right) + \left(\frac{R-r}{R+r}\right)^3\right] \\ = \frac{\pi R^3}{3(R+r)^3}[4r^3 + 12Rr^2] = \frac{4\pi R^2 r^2}{3(R+r)^3}[Rr + 3R^2]\end{aligned}$$

and the portion of the smaller sphere included within the frustum has volume

$$\frac{2\pi r^3}{3} + \pi\left[r^3\left(\frac{R-r}{R+r}\right) - \frac{r^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] = \frac{4\pi R^2 r^2}{3(R+r)^3}[Rr + 3r^2] \ .$$

Hence, the portions of the sphere lying within the frustum have total volume

$$\frac{4\pi R^2 r^2}{3(R+r)^3}[3R^2 + 2Rr + 3r^2] \ .$$

Subtracting this from the volume of the frustum yields the volume of the ring-shaped region

$$\frac{4\pi R^2 r^2}{3(R+r)^3}[(4R^2 + 4Rr + 4r^2) - (3R^2 + 2Rr + 3r^2)] = \frac{4\pi R^2 r^2}{3(R+r)^3}[R^2 + 2Rr + r^2] = \frac{4\pi R^2 r^2}{3(R+r)} \ .$$

Comment. The volume of a slice of a sphere of radius a and height h from the equatorial plane can be obtained from the volume of a right circular cone and a cylinder using the method of Cavalieri. The area of a cross-section of the slice at height t from the equator is $\pi(a^2 - t^2) = \pi a^2 - \pi t^2$. The term πa^2 represents the cross-section of a cylinder of radius a and height h while πt^2 represents the area of the cross section of a cone of base radius h at distance t from the vertex. Thus the area of the each cross-section of the cylinder is the sum of the areas of the corresponding cross-sections of the spherical slice and cone. Cavalieri's principle says that the volumes of the solids bear the same relation. Thus the volume of the spherical slice is

$$\pi a^2 h - \frac{1}{3}\pi h^3 \ .$$

636. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.

Solution. Let AD and BC intersect at P, $a_1 = |CP|$, $a_2 = |BP|$, $\alpha_1 = \angle CDP$, $\alpha_2 = \angle BDP$. Let BE and AC intersect at Q, $b_1 = |AQ|$, $b_2 = |CQ|$, $\beta_1 = \angle AEQ$, $\beta_2 = \angle CEQ$. Let CF and AB intersect at R, $c_1 = |BR|$, $c_2 = |AR|$, $\gamma_1 = \angle BFR$, $\gamma_2 = \angle AFR$.

Applying the Law of Sines to ΔBPD and ΔCPD, we find that

$$\frac{a_1}{\sin\alpha_1} = \frac{a_2}{\sin\alpha_2}$$

and similarly that

$$\frac{b_1}{\sin\beta_1} = \frac{b_2}{\sin\beta_2} \quad \text{and} \quad \frac{c_1}{\sin\gamma_1} = \frac{c_2}{\sin\gamma_2} \ .$$

Let $\alpha = \angle BAE$. Then $\alpha = \angle FAC$ since $\angle FAB = \angle EAC$. Similarly, let $\beta = \angle FBC = \angle ABD$ and $\gamma = \angle BCE = \angle ACD$.

Let $|AB| = c$, $|BC| = a$, $|AC| = b$, $|AD| = u$, $|BE| = v$, $|CF| = w$. By the Law of Sines, we find that

$$\frac{v}{\sin\alpha} = \frac{c}{\sin\beta_1} \quad \text{and} \quad \frac{v}{\sin\gamma} = \frac{a}{\sin\beta_2}$$

so that

$$\frac{c\sin\alpha}{\sin\beta_1} = \frac{a\sin\gamma}{\sin\beta_2} \Longrightarrow \frac{\sin\beta_1}{\sin\beta_2} = \frac{c}{a}\cdot\frac{\sin\alpha}{\sin\gamma} \quad .$$

Similarly

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{b}{c}\cdot\frac{\sin\gamma}{\sin\beta} \quad \text{and} \quad \frac{\sin\gamma_1}{\sin\gamma_2} = \frac{a}{b}\cdot\frac{\sin\beta}{\sin\alpha} \quad .$$

Putting this altogether yields

$$\frac{a_1}{a_2}\cdot\frac{b_1}{b_2}\cdot\frac{c_1}{c_2} = \frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}\cdot\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin\alpha}{\sin\gamma}\cdot\frac{\sin\beta}{\sin\alpha} = 1 \ .$$

By the converse of Ceva's Theorem, the cevians AP, BQ and CR are concurrent and the result follows.

637. Let n be a positive integer. Determine how many real numbers x with $1 \le x < n$ satisfy

$$x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3 \ .$$

Solution 1. Let $n - 1 \le x < n$. Then $\lfloor x^3 \rfloor = (n-1)^3 + r$ for $0 \le r < 3n(n-1)$. The equation is equivalent to

$$\lfloor x^3 \rfloor = \lfloor x \rfloor^3 + 3x\lfloor x \rfloor(x - \lfloor x \rfloor) = (n-1)^3 + 3x(n-1)(x-n+1) \ .$$

The increasing function $(n-1)^3 + 3x(n-1)(x-n+1)$ takes the value 0 when $x = n-1$ and $3n(n-1)$ when $x = n$. Therefore, on the interval $[n-1, n)$, it assumes each of the values $0, 1, \cdots, 3n(n-1)-1$ exactly once.

For $0 \le r < 3n(n-1)$, consider the equation

$$r = 3x(n-1)(x-n+1) \ .$$

This is equivalent to

$$\begin{aligned}(n-1)^3 + r &= (n-1)^3 - 3x(n-1)^2 + 3x^2(n-1) \\ &= [(n-1) - x]^3 + x^3 \ ,\end{aligned}$$

When x is a solution of this equation for which $n-1 \le x < n$, we have that $x^3 \le (n-1)^3 + r$ and

$$x^3 = (n-1)^3 + r + [x - (n-1)]^3 < (n-1)^3 + r + 1 \ ,$$

so that $\lfloor x^3 \rfloor = (n-1)^3 + r$¿ It follows that for each value of these values of r, the given equation is satisfied and so there are $3n(n-1)$ solutions x for which $n-1 \le x < n$.

Therefore, the total number of solutions not exceeding n is

$$\sum_{k=2}^{n} 3k(k-1) = \sum_{k=2}^{n} k^3 - (k-1)^3 - 1 = n^3 - 1 - (n-1) = n^3 - n \ .$$

Solution 2. Consider the behaviour of the two sides of the equation on the half-open interval defined by $k \le x < k+1$ for k a nonnegative integer. The function on the right increases continuously from 0 with right limit equal to 1. The function on the left increases continuously in the same way on each half-open interval defined by $\sqrt[3]{i} \le x < \sqrt[3]{i+1}$ for $k^3 \le i \le (k+1)^3 - 1 = k^3 + 3k(k+1)$. By examining the graphs, we see that they take equal values exactly once in each of the smaller intervals except the rightmost. Thus, they are equal $(k+1)^3 - k^3 - 1$ times. Therefore, over the whole of the interval defined by $1 \le x < n^3$, they are equal exactly

$$\sum_{k=1}^{n-1} [(k+1)^3 - k^3 - 1] = n^3 - 1^3 - (n-1) = n^3 - n$$

times, so that the given equation has this many solutions.

Solution 3. Let $x = k + r$, where k is a nonnegative integer and $0 \le r < 1$. Then

$$x^3 - \lfloor x^3 \rfloor = (k+r)^3 - (k^3 + \lfloor 3kr(k+r) + r^3 \rfloor)$$

so that the equation becomes

$$3kr(k+r) = \lfloor 3kr(k+r) + r^3 \rfloor \ .$$

This is equivalent to the assertion that $3kr(k+r)$ is an integer, so there is a solution to the equation for every x for which $3kr(k+r)$ is an integer, where $0 \le k \le n-1$ and $0 \le r < 1$.

Fix k. As r increases from 0 towards but not equal to 1, $3kr(k+r)$ increases from 0 up to but not including $3k(k+1)$, so it assumes exactly $3k(k+1)$ integer values. Hence the total number of solutions is

$$\sum_{k=0}^{n-1} 3k(k+1) = n^3 - n \ .$$

638. Let x and y be real numbers. Prove that

$$\max(0, -x) + \max(1, x, y) = \max(0, x - \max(1, y)) + \max(1, y, 1 - x, y - x)$$

where $\max(a, b)$ is the larger of the two numbers a and b.

Solution 1. [C. Deng] First, note that for real a, b, c, d,

$$\max(a, b) - c = \max(a - c, b - c) \ ;$$

$$\max(\max(a, b), c) = \max(a, b, c) \ ;$$

$$\max(a, b) + \max(c, d) = \max(a + c, a + d, b + c, b + d) \ .$$

[Establish these equations.] Then

$$\begin{aligned} \max(0, -x) &= \max(0, -x) + \max(1, y) - \max(1, y) \\ &= \max(1, y, 1 - x, y - x) - \max(1, y) \ ; \end{aligned}$$

and

$$\begin{aligned} \max(1, x, y) &= \max(1, x, y) - \max(1, y) + \max(1, y) \\ &= \max(\max(1, y), x) - \max(1, y) + \max(1, y) \\ &= \max(\max(1, y) - \max(1, y), x - \max(1, y)) + \max(1, y) \\ &= \max(0, x - \max(1, y)) + \max(1, y) \ . \end{aligned}$$

Adding these equations yields the desired result.

Solution 2. If $0 \le x \le 1$, then $-x \le 0$, $x - \max(1, y) \le x - 1 \le 0$, $1 - x \le 1$, $y - x \le y$, so that both sides are equal to $\max(1, y)$. If $x \le 0$, then $\max(0, -x) = -x$, $\max(1, x, y) = \max(1, y)$, $\max(0, x - \max(1, y)) = 0$ and $1 - x \ge 1$, $y - x \ge y$, so that

$$\max(1, y, 1 - x, y - x) = \max(1 - x, y - x) = \max(1, y) - x$$

which is the same as the left side.

Suppose that $x \geq 1$. Then the left side is equal to $0 + \max(x, y) = \max(x, y)$. When $y \leq 1$, the right side becomes $(x-1)+1 = x = \max(x, y)$. When $1 \leq y \leq x$, the right side becomes $x - y + y = x = \max(x, y)$. When $x \leq y$, the right side is $0 + y = \max(x, y)$. Thus, the result holds in all cases.

639. (a) Let $ABCDE$ be a convex pentagon such that $AB = BC$ and $\angle BCD = \angle EAB = 90°$. Let X be a point inside the pentagon such that AX is perpendicular to BE and CX is perpendicular to BD. Show that BX is perpendicular to DE.

(b) Let N be a regular nonagon, *i.e.*, a regular polygon with nine edges, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

(a) *Solution 1.* Let AX intersect BE in Y, CE intersect BD in Z and BX intersect DE in P. Assume X lies inside the triangle BDE; a similar proof holds when X lies outside the triangle BDE. From similar right triangles and since $AB = AC$, we have that

$$BY \cdot BE = AB^2 = AC^2 = BZ \cdot BD \ .$$

Hence triangles BYZ and BDE are similar and $\angle BYZ = \angle BDE$ and $\angle BZY = \angle BED$. Thus the quadrilateral $DEYZ$ is concyclic.

The quadrilateral $BYXZ$ is also concyclic, so that $\angle BZY = \angle BXY$. Therefore $\angle BED = \angle BXY$, with the result that triangles BXY and BEP are similar. Hence $\angle EPB = \angle XYB = 90°$.

Solution 2. [K. Zhou, J. Lei] Let T be selected on DE so that $BT \perp ED$. Let AY meet BT at S and CZ meet BT at R. Because triangles BSY and BET are similar, $BY : BR = BT : BE$, so that $BR \cdot BT = BY \cdot BE = AB^2$. Similarly, $BS \cdot BT = BZ \cdot BD = AC^2 = AB^2$. Hence $BR = BS$ so that $R = S$. So R and S must be the point X where AY and CZ meet and so T is none other than P. The result follows.

(b) *Answer*: $\angle OAB = 30°$.

Solution 1. [S. Sun] Let C be the point on OR for $BC \perp OR$. Since $\angle BOC = \angle QOA = 20°$, the right triangles BOC and QOA are similar, Since $QO = 2OB$, it follows that $AO = 2OC$.

Consider the triangle AOC. We have $AO = 2OC$ and $\angle AOC = 60°$. By splitting an equilateral triangle along a median, it is possible to construct a triangle UVW for which $AO = UV = 2VW$ and $\angle UVW = 60°$. Since also $VW = OC$, triangles AOC and UVW are congruent (SAS), so that $\angle OCA = \angle VWU = 90°$. Therefore, A, B, C are collinear, and $\angle OAB = \angle OAC = \angle UWV = 30°$.

Solution 2. Let C be the intersection of the radius perpendicular to QR and the circumcircle of N. We have that $\angle POQ = \angle QOR = 40°$. Thus, triangle OPC is equilateral, so that PB and OC are perpendicular. Since also $\angle OAP = 90°$, A and B lie on the circle with diameter OP, Hence $\angle OAB = \angle OPB = 30°$.

Solution 3. [D. Brox] $OA = r \sin 70°$ and $OD = \frac{r}{2} \cos 40°$, where r is the circumradius of the nonagon and D is the foot of the perpendicular from B to OA. Hence

$$AD = r(\sin 70° - \sin 30° \cos 40°) = r \sin 40° \cos 30° \ .$$

Therefore

$$\tan \angle OAB = \frac{BD}{AD} = \frac{OD \tan 40°}{AD} = \frac{\cos 40° \tan 40°}{2 \sin 40° \cos 30°} = \frac{1}{2\cos 30°} = \frac{1}{\sqrt{3}} \ ,$$

whence $\angle OAB = 30°$.

Solution 4. [H. Dong] Let E be the midpoint of OP so that triangle OEB is equilateral.

$$EB = EP \Longrightarrow \angle EPB = \angle EBP = 30° \Longrightarrow \angle OBP = 30° \ .$$

Hence $OBAP$ is concyclic, so that $\angle OAB = \angle OPB = 30°$.

Solution 5. [D. Arthur] $OB = \frac{1}{2}OP = OP\cos 60° = OP\cos\angle PQB$ so that $PB \perp OC$. Thus $OPAB$ is concyclic. Since $\angle OBA = 180° - \angle OPA = 180° - 70° = 110°$, then

$$\angle OAB = 180° - (\angle AOB + \angle OBA) = 180° - (40° + 110°) = 30° \ .$$

Solution 6. [F. Espinosa] $|\overrightarrow{OB}| = \frac{r}{2}$ and $|\overrightarrow{OA}| = r\cos 20°$. Then $\overrightarrow{OR} \cdot \overrightarrow{OB} = \frac{1}{2}r^2\cos 20°$ and $\overrightarrow{OR} \cdot \overrightarrow{OA} = r(r\cos 20°)\cos 60° = \frac{1}{2}r^2\cos 20°$. Hence $\overrightarrow{OR} \cdot \overrightarrow{AB} = overrightarrowOR \cdot \overrightarrow{OB} - overrightarrowOR \cdot \overrightarrow{OA} = 0$ with the result that $\angle ABO = 90°$. As before, it follows that $\angle OAB = 30°$.

Solution 7. [T. Costin] Let F be the midpoint of the side ST of the nonagon $PQRST\cdots$. Then $\angle AOF = 120°$, so $\angle OAG = 30°$ and $\angle OGA = 90°$, where G is the intersection point of AF and OR. Hence $OG = \frac{1}{2}OA$.

Let H be the intersection of AP and OC, with C the midpoint of RS. Then $OG = OH\cos 20°$. Also $OA = OQ\cos 20° = OR\cos 20°$. Hence

$$OH = \frac{OG}{\cos 20°} = \frac{OA}{2\cos 20°} = \frac{OR}{2}$$

so that $H = B$. Hence $\angle OAB = \angle OAH = 30°$.