PROBLEMS FOR NOVEMBER 2009

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

647. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

648. Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

649. In the triangle ABC, $\angle BAC = 20^\circ$ and $\angle ACB = 30^\circ$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10^\circ$. Determine $\angle BMC$.

650. Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} .$$

Determine the minimum value of

$$\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} .$$

651. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2 + 2y = x^3 - x^2 - x$ is satisfied by $(x, y) = (a(t)/c(t), b(t)/c(t))$.

652. (a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n + 1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m = 2$?

(c) Find all possible values of m that satisfy the condition in (a).

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n - r) : 1 \leq r \leq n - 1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a + b) = f(a) + f(b)$.