PROBLEMS FOR JUNE 2009

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download. The solutions will be published in a later Olymon.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

New procedure. Instead of the monthly problem sets with a long deadline, I plan to send out one or two problems a week which should be solved as soon as you can. I will record the problems in order of receipt and acknowledge solvers. Solutions will be published when there is no more activity on a problem. This month, I will pose the challenge problems for which solutions have been received to give anyone else a chance to solve them before solutions appear.

For those of you who are looking for practice problems, you can access old Olymon problems and solutions on the website sl www.math.utoronto.ca/barbeau/home.html or *www.cms.math.ca*; on the CMS website, you can also access International Mathematical Talent Search Problems as well as problems posed on the Canadian Open Mathematics Challenge and the Canadian Mathematical Olympiad.

Open Problems of the Week

626. [May 10-16.] Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

627. [May 17-23.] Let

$$f(x, y, z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a, b, c) = f(b, c, a) = f(c, a, b) \ .$$

Determine $f(a, b, c)$. Determine three such numbers a, b, c.

628. [May 24-30.] Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

629. [May 31-June 6] Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

630. [June 7-13] (a) Show that, if
$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \quad ,$$
then
$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

631. [June 14-20] The sequence of functions $\{P_n\}$ satisfies the following relations:
$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$
$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \quad , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

632. [June 21-27] Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a+b+c = x+y+z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.