Solutions for October

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are solutions to the equation for which the condition is satisfied and x, y, z are all positive.

Solution. [J. Zung] (a) The given equation is equivalent to $xy + yz + zx = a$, where $xyz \neq 0$. Observe that, if x, y and z are odd, then $xy + yz + zx$ is odd. Hence, when a is even, at least one of x, y and z is even and there are no solutions satisfying the coprimality condition.

Suppose that a is odd. Then all of its prime divisors exceed 2, so that a has no divisor save 1 in common with $ka - 1$ and $ka - 2$ for each integer k, so that

$$(x, y, z) = (ka - 1, -ka + 2, a + (ka - 1)(ka - 2))$$

is a solution of the desired type.

(b) Let p_i be the ith prime, for each positive integer i and let $a = p_1p_2 \cdots p_n - 1$. Then $xy+yz+zx = a$ has at least n solutions in positive integers, namely,

$$(x, y, z) = \left(1, p_i - 1, \frac{a+1}{p_i} - 1\right)$$

for $1 \leq i \leq n$. Let q be a prime divisor of a, Then q must exceed each p_i $(1 \leq i \leq n)$ and so cannot divide $p_i - 1$. Hence q cannot divide $a - (p_i - 1)$ and so not divide $(a - 1)/(p_i) - 1 = (a + 1 - p_i)/p_i$. Thus, for each of these solutions, gcd $(xyz, a) = 1$.

Comment. Another family of solutions for (a) is given by $(x, y, z) = (-k, k+1, a+k(k+1))$ for suitably chosen k. Let m be any number of the form $\prod(p - 1)^r$, where p is a prime divisor of a and r is a positive integer. (With a being odd, all such p are odd.) Since $2^{p-1} \equiv 1 \bmod p$ for any odd prime p, we can take $k = 2^m$. Then $k - 1$ is divisible by p for each prime divisor of a, so that k and $k+1$ cannot be divisible by p.

A. Abdi identified the solutions $(x, y, z) = (m, a + m^2 - m, 1 - m)$. When a is odd and a is a divisor of $m + 1$, then the greatest common divisor of a and xyz is 1.

An approach for (b) is to take $f(x, y, z) = xy + yz + zx$ and let a be any positive integer. Suppose that $a + 1 = (u + 1)(v + 1)$ for some positive integers u and v. Then $f(1, u, v) = u + v + uv = a$. If u and v are coprime, then a must be coprime to both u and v.

When a is even, no solution satisfying the requirements of the problem can be found in this way as both u and v must be even. However, if $a = 2m + 1$ is odd, we always have at least one positive solution, since $f(1, 1, m) = a$.

We might observe that the given equation is equivalent to $xy + yz + za = a$ or $a + z^2 = (x + z)(y + z)$, when $xyz \neq 0$. Thus, we can obtain infinitely many solutions to the equation by the following procedure. Given a, let z be arbitrary. Suppose $a + z^2 = uv$; then the equation is satified by $(x, y, z) = (u - z, v - z, z)$. We thus need to choose u and v carefully to adhere to the divisibility requirement. In (a), we take $u = 1$ and $v = a + z^2$; in (b), we take values of a so that u and v are large enough to make x and y both positive.

There are other ways of finding infinitely many a for which a positive solution exists. Suppose that $b \geq 4$ and $a = b^2 - 5$ and $z = 2$. Then $a + z^2 = b^2 - 1 = (b - 1)(b + 1)$. Therefore $(x, y, z) = (b - 3, b - 1, 2)$

satisfies the equation. Suppose that b is even and that p is a prime divisor of a. Then p is odd and $b^2 \equiv 5 \pmod{p}$. Since

$$(b-3)(b-1) = b^2 - 4b + 3 \equiv 8 - 4b = 4(2-b)$$

modulo p and since p cannot divide two of the consecutive integers $b-1$, $b-2$ and $b-3$, it follows that $(b-3)(b-1) \not\equiv 0$ (modulo p), so that the greatest common divisor of xyz and a is 1 as desired.

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB, for which the segments AU, BV, CW intersect in a common point O. Prove that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 ,$$

and

$$\frac{|AO|}{|OU|} \cdot \frac{|BO|}{|OV|} \cdot \frac{|CO|}{|OW|} = \frac{|AO|}{|OU|} + \frac{|BO|}{|OV|} + \frac{|CO|}{|OW|} + 2 .$$

Solution 1. Let F and G be points of BC for which $OF \| AB$ and $OG \| AC$. Then triangles BOG and BVC are similar, so that $|OV|/|BV| = |GC|/|BC|$. Similarly, $|OW|/|CW| = |BF|/|BC|$. Since triangles OFG and ABC are similar, $|OU|/|AU| = |FG|/|BC|$. Since $|BC| = |BF| + |FG| + |GC|$, adding the three equations yields that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1 .$$

Let $x = |AO|/|OU|$, $y = |BO|/|BV|$ and $z = |CO|/|CW|$. Then $|AU|/|OU| = 1+x$, $|BV|/|OV| = 1+y$ and $|CW|/|OW| = 1+z$, so that the foregoing equation can be rewritten

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1 .$$

Multiplying this equation by $(1+x)(1+y)(1+z)$ and simplifying yields that $xyz = x+y+z+2$ as desired.

Solution 2. Observe that $OU : AU = [OBC] : [ABC]$, $OV : BV = [OCA] : [ABC]$ and $OW : CW = [OAB] : [ABC]$ (where [] denotes area). Adding the three ratios gives the first result. If $|OU|/|AU| = u$, $|OV|/|BV| = v$ and $|OW|/|CW| = w$, then the left side of the second equation is

$$\begin{aligned}
\left(\frac{1}{u} - 1\right)\left(\frac{1}{v} - 1\right)\left(\frac{1}{w} - 1\right) &= \frac{1}{uvw}(1-u)(1-v)(1-w) \\
&= \frac{1 - (u+v+w) + (uv+vw+wu) - uvw}{uvw} \\
&= 0 + \frac{1}{u} + \frac{1}{v} + \frac{1}{w} - 1 \\
&= \left(\left(\frac{1}{u} - 1\right)\left(\frac{1}{v} - 1\right)\left(\frac{1}{w} - 1\right) + 2 .
\end{aligned}$$

Solution 3. [D. Hidru] We can assign weights v and w to points B and C so that $w : v = CV : BV$ and a weight u to A so that $v : u = AW : BW$. The centre of gravity of v and w is at U and of v and u is at W. The centre of gravity of u, v, w is at O, the common point of AU and CW, so that the centre of gravity of u and w lies at V, the common point of BO and AC. Since O is the centre of gravity of weight u at A and $v+w$ at U, $AO : OU = (b+c) : a \Rightarrow OU : AU = a : (a+b+c)$. Similarly, $BO : OV = (a+c) : b \Rightarrow OV : BV = b : a+b+c$ and $CO : OW = (a+b) : c \Rightarrow OW : CW = c : (a+b+c)$.

The left side of the first equation is

$$\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1 ,$$

while the left side of the second equation is

$$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) = \frac{1}{abc}[2abc + (a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)]$$
$$= 2 + \frac{ab(a+b)}{abc} + \frac{bc(b+c)}{abc} + \frac{ca(c+a)}{abc} = 2 + \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \ .$$

The desired results hold.

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD. Prove that one of the triangles MON and LOK is a $90°$ rotation of the other around O.

What happens when $ABCD$ is a parallelogram?

Solution 1. [A. Abdi] We establish a lemma: *Let X, Y be points external to triangle PQR such that triangle PXQ and PYR are isosceles with right angles at X and Y. Let Z be the midpoint of QR. Then the segments YZ and XZ are equal and perpendicular.*

To prove this, let S and T be the respective midpoints of PQ and PR. Note that $SZ \| PR$ and $TZ \| PQ$. Then

$$\angle XSZ = 90° + \angle QSZ = 90° + \angle QPR = 90° + \angle ZTR = \angle ZTY \ ,$$

$SZ = TR = TY$ and $XS = QS = ZT$. Therefore triangles XSZ and ZTY are congruent (SAS), so that $XZ = ZY$.

Triangle XSZ is transformed to triangle ZTY by the composite of a $90°$ rotation about S that takes $S \rightarrow S$, $X \rightarrow Q$, $Z \rightarrow Z'$, and a translation in the direction of PQ that takes $S \rightarrow T$, $Q \rightarrow Z$, $Z' \rightarrow Y$. Hence $YZ \perp XZ$.

Apply this to the situation at hand. With regard to triangle ABD, we find that OK is equal and perpendicular to ON. With respect to triangle BCD, we find that OL is equal and perpendicular to OM. Therefore a $90°$ rotation about O takes $K \rightarrow N$ and $L \rightarrow M$, and the result follows.

Solution 2. A $90°$ rotation about M that takes B to C takes O to a point O' and C to a point C'. A $90°$ degree rotation about L that takes D to C takes O to a point O'' and C to a point C''. We have that

$$|CO'| = |BO| = |DO| = |CO''|$$

and $CO' \perp BO$, $CO'' \perp DO$. Therefore $O' = O''$. Since $MO = MO'$, $MO \perp MO'$, $LO = LO''$ and $LO \perp LO''$, OMO' and $OLO'' = OLO'$ are two isosceles right triangles with a common hypotenuse OO'. Therefore $LO = MO$ and $\angle LOM = 90°$.

Similarly, $OK = ON$ and $OK \perp ON$, from which the result follows.

If $ABCD$ is a parallelogram, then O is the centre of a half turn that interchanges B and D, A and C, M and N, as well as L and K, so that MON and LOK are both straight lines.

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60°$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

Solution. Let XY and ZY be two rays that meet at Y at an angle of $60°$. On the ray YX, locate points P, Q, R so that $|YP| = |OA|$, $|YQ| = |OC|$ and $|YR| = |OE|$. Similarly, on the ray YZ, locate points U, V, W so that $|YU| = |OB|$, $|YV| = |OD|$ and $|YW| = |OF|$. Then we have (by SAS congruency) that $|AB| = |PU|$, $|BC| = |UQ|$, $|CD| = |QV|$, $|DE| = |VR|$, $|EF| = |RW|$ and $|FA| = |WP|$.

Suppose that PW intersects QU and RV in the respective points S and T. Then

$$\begin{aligned}|AB| + |CD| + |EF| &= |PU| + |QV| + |RW| \\ &< (|PS| + |SU|) + (|QS + |ST| + |TV|) + (|RT| + |TW|) \\ &= |PS| + |QU| + |ST| + |RV| + |TW| = |QU| + |RV| + |PW| \\ &= |BC| + |DE| + |FA| \ .\end{aligned}$$

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

Solution 1. Let E_n be the expected number of tosses, when the maximum number of tosses is n. When $n = 1, 2$, there must be n tosses, so that $E_1 = 1$ and $E_2 = 2$. (When $n = 3$, the only occurrence with fewer than three tosses is HT, which occurs with probability $\frac{1}{4}$, so that $E_3 = 11/4$. Similarly, $E_4 = 27/8$.)

Let $n \geq 3$, and consider the case of a maximum of n tosses. There are three mutually exclusive events:
(1) The tossing begins with a tail, with probability 1/2. The expected number of tosses is then $1 + E_{n-1}$;
(2) The tossing begins with two heads, with probability 1/4. The expected number of tosses is then $2+E_{n-2}$;
(3) The tossing begins with a head followed by a tail, with probability 1/4, at which point the tossing stops.

Thus,

$$E_n = \frac{1}{2}(1 + E_{n-1}) + \frac{1}{4}(2 + E_{n-2}) + \frac{1}{4}2 = \frac{3}{2} + \frac{1}{2}E_{n-1} + \frac{1}{4}E_{n-2} \ .$$

One solution of the recursion is $E_n \equiv 6$ for all n. This is not the solution we are seeking. Set $X_n = 2^n(E_n - 6)$. Then $X_n = X_{n-1} + X_{n-2}$, for $n \geq 3$. The initial conditions are $X_1 = -10 = -2f_5$ and $X_2 = -16 = -2f_6$, where $\{f_n : n \geq 1\} = \{1, 1, 2, 3, 5, 8, \cdots\}$ is the Fibonacci sequence with $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Therefore

$$E_n = 6 + \frac{X_n}{2^n} = 6 - \frac{f_{n+4}}{2^{n-2}} \ .$$

Comment. More basically, the argument setting up the recursion for E_n is the following. Let p_k be the probability that k tosses are made in which the first odd number sequence of heads is concluded by the last toss, which is a tail and q_k the probability that k tosses are made without having to stop earlier at the termination by a tail of an odd number of heads. Then, for each $n \geq 1$, $E_n = \sum_{k=1}^{n-1} kp_k + nq_n$, where $\sum_{k=1}^{n-1} p_k + q_n = 1$. Then

$$\begin{aligned}E_n &= \frac{1}{2}\left[\sum_{k=1}^{n-2}(1+k)p_k + (1+n-1)q_{n-1}\right] + \frac{1}{4}\left[\sum_{k=1}^{n-3}(2+k)p_k + (2+n-2)q_{n-2}\right] + \frac{1}{4}[2] \\ &= \frac{1}{2}\left[\left(\sum_{k=1}^{n-2} p_k\right) + q_{n-1} + \sum_{k=1}^{n-2} kp_k + (n-1)q_{n-1}\right] \\ &\qquad + \frac{1}{4}\left[2\left(\sum_{k=1}^{n-3} p_k + q_{n-2}\right) + \sum_{k=1}^{n-3} kp_k + (n-2)q_{n-2}\right] + \frac{1}{2} \\ &= \frac{1}{2}[1 + E_{n-1}] + \frac{1}{4}[2 + E_{n-2}] + \frac{1}{2} \ .\end{aligned}$$

Solution 2. [J. Zung] As in Solution 1, we can derive $E_n = \frac{1}{2}E_{n-1} + \frac{1}{4}E_{n-2} + \frac{3}{2}$ for $n \geq 2$, where $E_0 = 0$. Let

$$f(x) = \sum_{n=1}^{\infty} E_n x^n$$

be the generating function for E_n. Then

$$\begin{aligned}
f(x) &= x + \sum_{n=2}^{\infty} E_n x^n \\
&= x + \frac{1}{2}\sum_{n=2}^{\infty} E_{n-1}x^n + \frac{1}{4}\sum_{n=2}^{\infty} E_{n-2}x^n + \frac{3}{2}\sum_{n=2}^{\infty} x^n \\
&= x + \frac{x}{2}f(x) + \frac{x^2}{4}f(x) + \frac{3x^2}{2(1-x)} \\
&= \frac{x}{2}f(x) + \frac{x^2}{4}f(x) + \frac{x(x+2)}{2(1-x)} \ .
\end{aligned}$$

Therefore

$$\begin{aligned}
f(x) &= \frac{x(x+2)}{2(1-x)(1-\frac{1}{2}x-\frac{1}{4}x^2)} \\
&= \frac{x(x+2)}{2(1-x)(1-\alpha x)(1-\beta x)} \\
&= \frac{C}{1-x} + \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\
&= \sum_{n=0}^{\infty}(C + A\alpha^n + B\beta^n)x^n \ ,
\end{aligned}$$

where $\alpha+\beta = \frac{1}{2}$, $\alpha\beta = -\frac{1}{4}$, so that $\alpha = \frac{1}{4}(1+\sqrt{5})$, $\beta = \frac{1}{4}(1-\sqrt{5})$, and where A, B, C satisfy

$$\frac{1}{2}x(x+2) = A(1-x)(1-\beta x) + B(1-x)(1-\alpha x) + C(1-\alpha x)(1-\beta x) \ .$$

Comparing leading and constant coefficients leads to

$$\frac{1}{2} = A\beta + B\alpha + C\alpha\beta = A\beta + B\alpha - \frac{1}{4}C$$

and

$$0 = A + B + C \ .$$

Substituting $x = 1$ leads to

$$\frac{3}{2} = C(1-\alpha)(1-\beta) = C\left(1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{C}{4} \ .$$

Therefore $C = 6$ and $2 = A\beta + B\alpha = A\beta - (6+A)\alpha$. Thus we find that

$$A = -\frac{1}{5}(15 + 7\sqrt{5})$$

and

$$B = -\frac{1}{5}(15 - 7\sqrt{5}) \ .$$

Hence,

$$f(x) = \sum_{n=0}^{\infty}\left[6 - \frac{1}{5}(15+7\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)^n - \frac{1}{5}(15-7\sqrt{5})\left(\frac{1-\sqrt{5}}{4}\right)^n\right]x^n \ .$$

Therefore

$$E_n = 6 - \frac{1}{5}(15+7\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)^n - \frac{1}{5}(15-7\sqrt{5})\left(\frac{1-\sqrt{5}}{4}\right)^n \ .$$

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n-1)$.

Solution 1. Define the function $p(n)$ to be 0 when $n < 0$, 1 when $n = 0$ and the number of partitions of n when $n > 0$. For example, $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$. For each positive integer i, $p(n-2i)$ is the number of partitions of n that have at least i occurrences of 2. Suppose that a partition has exactly k occurrences of 2; then this partition is counted once among $p(n-2i)$ for $1 \leq i \leq k$. Therefore

$$f(n) = p(n-2) + p(n-4) + p(n-6) + \cdots = \sum_{i=1}^{\infty} p(n-2i) \ .$$

The total number of partitions of n in which a number j occurs as a singleton is $p(n-j) - p(n-2j)$. Hence

$$\begin{aligned} g(n) &= [p(n-1) - p(n-2)] + [p(n-2) - p(n-4)] + [p(n-3) - p(n-6)] + \cdots \\ &= p(n-1) + p(n-3) + p(n-5) + \cdots = \sum_{j=1}^{\infty} p(n+1-2j) \ . \end{aligned}$$

It follows that $f(n) = g(n-1)$.

Solution 2. [A. Abdi] Let $u(n,i)$ be the number of partitions of n for which the number 2 appears exactly i times, and let $v(n,j)$ be the number of partitions of n that have exactly one occurrence of j. Then

$$f(n) = \sum_{i=1}^{n} iu(n,i)$$

and

$$g(n) = \sum_{j=1}^{n} v(n,j) \ .$$

Since the number of partitions with exactly i occurrences of 2 is equal to the number with at least i minus the number with at least $i+1$ occurrences, we have that $u(n,i) = p(n-2i) - p(n-2i-2)$, whence

$$\begin{aligned} f(n) &= \sum_{i=1}^{n} i(p(n-2i) - p(n-2(i+1)) \\ &= p(n-2) - p(n-4) + 2p(n-4) - 2p(n-6) + 3p(n-6) - p(n-8) + \cdots \\ &= \sum_{i=1}^{n} p(n-2i) \ . \end{aligned}$$

Since the number of partitions with exactly one occurrence of j is equal to the number with at least one occurrence minus the number with at least two occurrences, we have that $v(n,j) = p(n-j) - p(n-2j)$, whence

$$\begin{aligned} g(n) &= [p(n-1) - p(n-2)] + [p(n-2) - p(n-4)] + [p(n-3) - p(n-6) + \cdots \\ &= \sum_{j=1}^{n} p(n+1-2j) \ . \end{aligned}$$

The desired result follows.

Solution 3. The generating function for $p(n)$ is

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \ .$$

The number of partitions of n with exactly k occurrences of 2 is the coefficients of x^n in $x^{2k}(1-x^2)P(x)$. Therefore the total number of 2s in all partitions of n is the coefficient of x^n in

$$\sum_{k=1}^{\infty} kx^{2k}(1-x^2)P(x) = \left(\frac{x^2}{(1-x^2)^2}\right)(1-x^2)P(x) = \frac{x^2}{1-x^2}P(x) \ .$$

The number of partitions of $n-1$ containing k as a singleton is the coefficient of x^n in $x^k(1-x^k)P(x)$. Therefore the number of singletons over all partitions of $n-1$ is the coefficient of x^n in

$$x\sum_{k=1}^{\infty} x^k(1-x^k)P(x) = x\left(\frac{x}{1-x} - \frac{x^2}{1-x^2}\right)P(x) = \frac{x^2}{1-x^2}P(x) \ .$$

The result follows.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b.

(b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d

Solution. (a) Let the centres of the three circles with radii a, b, c be respectively A, B, C and let the points of tangency with the line be respectively U, V, W. Then

$$|UV| = \sqrt{(a+b)^2 - (a-b)^2} = 2\sqrt{ab} \ ,$$

$$|UW| = 2\sqrt{ac} \qquad \text{and} \qquad |VW| = 2\sqrt{bc} \ .$$

Hence $\sqrt{ab} = \sqrt{ab} + \sqrt{bc}$, so that

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \ ,$$

or

$$c = \frac{ab}{a+b+2\sqrt{ab}} \ .$$

(b) Let the centres of the four circles with radii a, b, c, d be respectively A, B, C, D, and let $\mathbf{u} = \overrightarrow{DA}$, $\mathbf{v} = \overrightarrow{DB}$ and $\mathbf{w} = \overrightarrow{DC}$ denote the vectors from the centre D to the other three centres. Up to labelling of the circles, there are two possible configurations to consider. The first is that the circle of centre D contains the other three circles, in which case $\mathbf{u} \cdot \mathbf{u} = (d-a)^2 = (a-d)^2$, $\mathbf{v} \cdot \mathbf{v} = (d-b)^2 = (b-d)^2$, $\mathbf{w} \cdot \mathbf{w} = (d-c)^2 = (c-d)^2$. The second is that in which the circle of centre D is exterior to and surrounded by the three other circles. In this case, $\mathbf{u} \cdot \mathbf{u} = (d+a)^2 = (a+d)^2$, $\mathbf{v} \cdot \mathbf{v} = (b+d)^2$ and $\mathbf{w} \cdot \mathbf{w} = (c+d)^2$. We can comprise these two cases into one by adopting the convention that the in the first case, with the largest circle containing the others, the radius of the largest circle is assigned a negative value. Let $p = 1/a$, $q = 1/b$, $r = 1/c$ and $s = 1/d$ be the curvatures of the circle, where s can be positive or negative depending on the configuration. For both configurations, we have that

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = |AB|^2 = (a+b)^2 \ ,$$

$$(\mathbf{v}-\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = |BC|^2 = (b+c)^2 \ ,$$
$$(\mathbf{w}-\mathbf{u})\cdot(\mathbf{w}-\mathbf{u}) = |AC|^2 = (c+a)^2 \ .$$

The three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ reside in two-dimensional space; therefore, they are linearly dependent. This means that there are constants, α, β, γ, not all zero, for which

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = 0 \ .$$

Taking the inner product of each equation with the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we find that the following system of three equations in α, β, γ has a nontrivial solution:

$$(\mathbf{u}\cdot\mathbf{u})\alpha + (\mathbf{u}\cdot\mathbf{v})\beta + (\mathbf{u}\cdot\mathbf{w})\gamma = 0 \ ,$$
$$(\mathbf{u}\cdot\mathbf{v})\alpha + (\mathbf{v}\cdot\mathbf{v})\beta + (\mathbf{v}\cdot\mathbf{w})\gamma = 0 \ ,$$
$$(\mathbf{u}\cdot\mathbf{w})\alpha + (\mathbf{v}\cdot\mathbf{w})\beta + (\mathbf{w}\cdot\mathbf{w})\gamma = 0 \ .$$

From the first two equations, we find that

$$\alpha : \beta : \gamma = [(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) - (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})] : [(\mathbf{u}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{w}] : [(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2] \ .$$

Plugging this into the third equation yields that

$$2(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{w}) + (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) = (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{w})^2 + (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w})^2 + (\mathbf{w}\cdot\mathbf{w})(\mathbf{u}\cdot\mathbf{v})^2 \ . \qquad (1)$$

Recall that $\mathbf{u}\cdot\mathbf{u} = (a+d)^2$, $\mathbf{v}\cdot\mathbf{v} = (b+d)^2$ and $\mathbf{w}\cdot\mathbf{w} = (c+d)^2$. Since

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = |AB|^2 = (a+b)^2 \ ,$$

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}[(a+d)^2 + (b+d)^2 - (a+b)^2] = d^2 + ad + bd - ab = (a+d)(b+d) - 2ab \ .$$

Similarly,

$$\mathbf{u}\cdot\mathbf{w} = (a+d)(c+d) - 2ac \ ,$$

and

$$\mathbf{v}\cdot\mathbf{w} = (b+d)(c+d) - 2bc \ .$$

Expressing these in terms of p, q, r, s, we find that

$$\mathbf{u}\cdot\mathbf{u} = \frac{(p+s)^2}{p^2s^2} \ , \qquad \mathbf{v}\cdot\mathbf{v} = \frac{(q+s)^2}{q^2s^2} \ , \qquad \mathbf{w}\cdot\mathbf{w} = \frac{(r+s)^2}{r^2s^2} \ ,$$

$$\mathbf{u}\cdot\mathbf{v} = \frac{(p+s)(q+s) - 2s^2}{pqs^2} \ ,$$

$$\mathbf{v}\cdot\mathbf{w} = \frac{(q+s)(r+s) - 2s^2}{qrs^2} \ ,$$

$$\mathbf{w}\cdot\mathbf{u} = \frac{(r+s)(p+s) - 2s^2}{prs^2} \ .$$

The left side of (1), multiplied by $p^2q^2r^2s^6$, is equal to

$$\begin{aligned} 2(p+s)^2(q+s)^2)r+s)^2 - 4s^2(p+s)(q+s)(r+s)[(p+s)+(q+s)+(r+s)] \\ + 8s^4[(p+s)(q+s)+(q+s)(r+s)+(r+s)(p+s)] \\ - 16s^6 + (p+s)^2(q+s)^2(r+s)^2 \ ; \end{aligned}$$

the right side of (1), multiplied by $p^2q^2r^2s^6$ is equal to

$$(p+s)^2[(q+s)^2(r+s)^2 - 4s^2(q+s)(r+s) + 4s^4] + (q+s)^2[(r+s)^2(p+s)^2 - 4s^2(r+s)(p+s) + 4s^4] + (r+s)^2[(p+s)^2(q+s)^2 - 4s^2(p+s)(q+s) + 4s^4] .$$

Equating the two sides, removing common terms and dividing by $4s^4$ yields the equation

$$2[(p+s)(q+s) + (q+s)(r+s) + (r+s)(p+s)] - 4s^2 = (p+s)^2 + (q+s)^2 + (r+s)^2$$

$$\Longrightarrow 2(pq+qr+rp) + 4s(p+q+r) + 2s^2 = p^2+q^2+r^2+2s(p+q+r)+3s^2$$

$$\Longrightarrow 2(pq+qr+rp+ps+qs+rs) = p^2+q^2+r^2+s^2$$

$$\Longrightarrow (p+q+r+s)^2 = 2(p^2+q^2+r^2+s^2)$$

$$\Longrightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 = 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right) .$$

Solution 2. [J. Zung] Suppose that the centres and respective radii of the four circles are A, B, C, D and a, b, c, d. Consider the case that the circle with centre D is surrounded by the other three circles. Let $p = 1/a$, $q = 1/b$, $r = 1/c$ and $s = 1/d$. Let the incircles of the triangles ABC, ABD, ACD and BCD be centred at D', C', B' and A' and their radii be d', c', b' and a' respectively. Let $p' = 1/a'$, $q' = 1/b'$, $r' = 1/c'$ and $s' = 1/d'$.

Observe that the circle of centre D' intersects the circles of centres A and B at their point of tangency, the circles of centres B and C at their point of tangency and the circles of centres A and C at their point of tangency. (Why?) The sides of triangle ABC are $a+b$, $a+c$, $b+c$, the semi-perimeter is $a+b+c$ and the inradius is d'. Equating two determinations of the area of triangle $[ABC]$ yields the equation

$$d'(a+b+c) = \sqrt{abc(a+b+c)} ,$$

whence

$$d' = \sqrt{\frac{abc}{a+b+c}} \Longrightarrow \frac{1}{d'^2} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$$

and

$$s'^2 = pq + qr + rp .$$

Similarly, we obtain that

$$p'^2 = qr + rs + sq ; \quad q'^2 = pr + rs + sp ; \quad r'^2 = pq + qs + sp .$$

Adding these four equations together yields that

$$p'^2 + q'^2 + r'^2 + s'^2 = (p+q+r+s)^2 - (p^2+q^2+r^2+s^2) . \qquad (1)$$

Observe that $A'B'$ is tangent to the circle of centre D, *etc.*, so that this circle is the incentre of triangle $A'B'C'$. Equating two expressions for the area of triangle $A'B'C'$ yields the equation

$$s^2 = p'q' + q'r' + r's' .$$

The circle with centre A and radius a is the escribed circle opposite D' and tangent to $B'C'$ of triangle $B'C'D'$, whose sides are of lengths $d'-c'$, $d'-b'$ and $b'+c'$ and whose semiperimeter is d'. Thus, $[B'C'D'] = a(d'-b'-c') = \sqrt{(d'-b'-b')d'b'c'}$ so that

$$\frac{1}{a^2} = \frac{1}{b'c'} - \frac{1}{b'd'} - \frac{1}{c'd'} \Longrightarrow p^2 = q'r' - q's' - r's' .$$

Similarly,

$$q^2 = p'r' - p's' - q's' \quad \text{and} \quad r^2 = p'q' - p's' - q's' \ .$$

Therefore

$$p^2 + q^2 + r^2 + s^2 = (p' + q' + r' - s')^2 - (p'^2 + q'^2 + r'^2 + s'^2) \ . \tag{2}$$

Comparing equations (1) and (2), we conclude that

$$p' + q' + r' - s' = p + q + r + s \ . \tag{3}$$

(Why is the left side positive?) Call the common value σ. Then, using either (1) or (2),

$$\begin{aligned}\sigma^2 &= (p+q+r+s)^2 = (p^2+q^2+r^2+s^2) + (p'^2+q'^2+r'^2+s'^2) \\ &= (p+q+r)^2 - 2(pq+qr+rp) + s^2 + (p'+q'+r')^2 - 2(p'q'+q'r'+r's') + s'^2 \\ &= (\sigma - s)^2 - 2s'^2 + s^2 + (\sigma + s')^2 - 2s^2 + s'^2 \\ &= 2\sigma^2 - 2\sigma(s - s') \ ,\end{aligned}$$

whence $\sigma = 2(s - s')$. Similarly, it can be shown that $\sigma = 2(a + a') = 2(b + b') = 2(c + c')$. Since, from (3),

$$2\sigma[(p - p') + (q - q') + (r - r') + (s + s')] = 0 \ ,$$

we can make the various substitutions for σ to obtain that

$$p^2 + q^2 + r^2 + s^2 = p'^2 + q'^2 + r'^2 + s'^2 = (p+q+r+s)^2 - (p^2+q^2+r^2+s^2 \ .$$

from which

$$2(p^2 + q^2 + r^2 + s^2) = (p + q + r + s)^2 \ .$$

A similar analysis can be made to the same result when the circles with centres A, B and C are contained within the circle of centre D. When one of the circles, say of centre D, is replaced by a straight line, then $s = 0$ and the condition becomes $2(p^2 + q^2 + r^2) = (p + q + r)^2$.

Comment. In the special case that three of the circles have radius 1, say $p = q = r = 1$, then it can be checked directly that the radius of the inner fourth circle is $\frac{1}{3}(2\sqrt{3} - 3)$ and of the outer fourth circle is $-\frac{1}{3}(2\sqrt{3} + 3)$. The quadratic equation to be solved for s is $s^2 - 6s - 3 = 0$, whose roots are $s = 3 \pm 2\sqrt{3} = 3(-3 \pm 2\sqrt{3})^{-1}$.

Part (a) can be considered a special case of (b), where the fourth circle has infinite radius and curvature 0. In this case, $s = 0$, and the condition in (a) reads

$$\begin{aligned}\sqrt{r} = \sqrt{p} + \sqrt{r} &\Longrightarrow r - p - q = 2\sqrt{pq} \\ &\Longrightarrow p^2 + q^2 + r^2 - 2pr - 2qr + 2pq = 4pq \\ &= 2(p^2 + q^2 + r^2) = (p + q + r)^2 \ ,\end{aligned}$$

The result (b) is quite ancient. Known as the Descartes' Circle Theorem, it was disclosed by him in a letter to Princess Elisabeth of Bohemia in November, 1643, although his proof is unclear. It was rediscovered in 1842 by Philip Beecroft and again in 1936 by Frederick Soddy, a physicist and Nobel prizewinner, who published notes in *Nature* magazine (137 (1936), 1021; 139 (1939), 62). This result is discussed by H.S.M. Coxeter on pages 11-16 of his *Introduction to geometry, second edition* (Wiley, 1961, 1969). Other treatments and extensions appear in the *American Mathematical Monthly* (Vandeghen, 71 (1964), 176-179; Alexander, 74 (1967), 128-140; Pedoe, 74 (1967), 627-640; Coxeter, 75 (1968), 5-15; Brown, 76 (1969), 661-663; Lagarias, 109 (2002), 338-361). The result can be extended to spheres of higher dimension.