Solutions for December

472. Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

Solution. If $x < 0$, then the left side is an integer, but the right side is positive and less than $\frac{1}{4} + \frac{1}{5} < 1$. If $x > 5$, then the left side is less than $\frac{1}{4}$, while the right side is a positive integer. Therefore, the only candidates for solution are the integers between 0 and 5 inclusive. Checking, we find that the only solution is $x = 2$.

473. Let $ABCD$ be a quadrilateral; let M and N be the respective midpoint of AB and BC; let P be the point of interesection of AN and BD, and Q be the point of intersection of DM amd AC. Suppose the $3BP = BD$ and $3AQ = AC$. Prove that $ABCD$ is a parallelogram.

Solution. Let $\overrightarrow{AB} = \mathbf{x}$, $\overrightarrow{BC} = \mathbf{y}$ and $\overrightarrow{CD} = a\mathbf{x} + b\mathbf{y}$, where a and b are real numbers. Then

$$\overrightarrow{AD} = (a+1)\mathbf{x} + (b+1)\mathbf{y}$$

and

$$\overrightarrow{AN} = \mathbf{x} + \frac{1}{2}\mathbf{y} \ .$$

But $\overrightarrow{BD} = 3\overrightarrow{BP}$, so that

$$\overrightarrow{AP} = \frac{2\overrightarrow{AB} + \overrightarrow{AD}}{3} = \frac{a+3}{3}\mathbf{x} + \frac{b+1}{3}\mathbf{y} \ .$$

Since the vectors $\overrightarrow{AP}$ and $\overrightarrow{AN}$ are collinear, $a+3 : 1 = b+1 : \frac{1}{2}$, whence $a - 2b + 1 = 0$. Also

$$\overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD} = \left(\frac{1}{2} - a - 1\right)\mathbf{x} - (b+1)\mathbf{y} = -\left(a + \frac{1}{2}\right)\mathbf{x} - (b+1)\mathbf{y}$$

and

$$\overrightarrow{DQ} = \overrightarrow{AQ} - \overrightarrow{AD} = \frac{1}{3}(\mathbf{x}+\mathbf{y}) - (a+1)\mathbf{x} - (b+1)\mathbf{y} = -\frac{1}{3}[(3a+2)\mathbf{x} + (3b+2)\mathbf{y}] \ .$$

Since the vectors $\overrightarrow{DQ}$ and $\overrightarrow{DM}$ are collinear, we must have $(3a+2) : (a+\frac{1}{2}) = (3b+2) : (b+1)$, whence $2a + b + 2 = 0$. Therefore $(a,b) = (-1,0)$, $\overrightarrow{CD} = -\mathbf{x} = \overrightarrow{BA}$ and $\overrightarrow{AD} = \mathbf{y} = \overrightarrow{BC}$. Hence $ABCD$ is a parallelogram.

474. Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

Solution. Recall the identity $u^{\log_b v} = v^{\log_b u}$ for positive u, v and positive base $b \neq 1$. (Take logarithms to base b.) Then, for all real t, $(2^t + 3)^{\log_5 2} = 2^{\log_5(2^t+3)}$. This is true in particular when $t = \log_5 x$.

Let $f(x) = 2^{\log_5 x} + 3$ for $x > 0$. Then $f(x) = x^{\log_5 2} + 3$ and the equation to be solved is $f(f(x)) = x$. The function $f(x)$ is an increasing function of the positive variable x. If $f(x) < x$, then $f(f(x)) < f(x)$; if $f(x) > x$, then $f(f(x)) > f(x)$. Hence, for $f(f(x)) = x$ to be true, we must have $f(x) = x$. With $t = \log_5 x$, the equation becomes $2^t + 3 = 5^t$, or equivalently, $(2/5)^t + 3(1/5)^t = 1$. The left side is a stricly decreasing function of t, and so equals the right side only when $t = 1$. Hence the unique solution of the equation is $x = 5$.

475. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

Solution. Let $s = z_1 + z_2 + z_3 + z_4$. Then

$$|s - (1-t)z_1| = |s - (1-t)z_2| = |s - (1-t)z_3| \ .$$

Therefore, s is equidistant from the three distinct points $(1-t)z_1$, $(1-t)z_2$ and $(1-t)z_3$; but these three points are on the circle with centre 0 and radius $(1-t)z_1$. Therefore $s = 0$.

Since $z_1 - (-z_2) = z_1 + z_2 = -z_3 - z_4 = (-z_4) - z_3$ and $z_2 - (-z_3) = z_2 + z_3 = -z_4 - z_1 = (-z_4) - z_1$, z_1, $-z_2$, z_3 and $-z_4$ are the vertices of a parallelogram inscribed in a circle centered at 0, and hence of a rectangle whose diagonals intersect at 0. Therefore, $-z_2$ is the opposite of one of z_1, z_3 and $-z_4$. Since z_2 is unequal to z_1 and z_3, we must have that $-z_2 = z_4$. Also $z_1 = -z_3$. Hence z_1, z_2, z_3 and z_4 are the vertices of a rectangle.

476. Let p be a positive real number and let $|x_0| \leq 2p$. For $n \geq 1$, define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 \ .$$

Determine x_n as a function of n and x_0.

Solution. Let $x_n = 2py_n$ for each nonnegative integer n. Then $|y_0| \leq 1$ and $y_n = 3y_{n-1} - 4y_{n-1}^3$. Recall that

$$\sin 3\theta = \sin 2\theta \cos\theta + \sin\theta \cos 2\theta = 2\sin\theta(1 - \sin^2\theta) + \sin\theta(1 - 2\sin^2\theta) = 3\sin\theta - 4\sin^3\theta \ .$$

Select $\theta \in [-\pi/2, \pi/2]$. Then, by induction, we determine that $y_n = \sin 3^n\theta$ and $x_n = 2p\sin 3^n\theta$, for each nonnegative integer n, where $\theta = \arcsin(x_0/2p)$.

477. Let S consist of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Find all functions that map S into the set $\mathbf{R}$ of reals such that (1) f is increasing, and (2) $f(x+y) = f(x) + f(y)$ for all x, y in S.

Solution. Since $f(0) = f(0) + f(0)$, $f(0) = 0$ and $f(x) \geq 0$ for $x \geq 0$. Let $f(1) = u$ and $f(\sqrt{2}) = v$; u and v are both nonnegative. Since $f(0) = f(x) + f(-x)$, $f(-x) = -f(x)$ for all x. Since, by induction, it can be shown that $f(nx) = nf(x)$ for every positive integer n, it follows that

$$f(a + b\sqrt{2}) = au + bv \ ,$$

for every pair (a, b) of integers.

Since f is increasing, for every positive integer n, we have that

$$f(\lfloor n\sqrt{2} \rfloor) \leq f(n\sqrt{2}) \leq f(\lfloor n\sqrt{2} \rfloor + 1) \ ,$$

so that

$$\lfloor n\sqrt{2} \rfloor u \leq nv \leq (\lfloor n\sqrt{2} \rfloor + 1)u \ .$$

Therefore,

$$\left(\sqrt{2} - \frac{1}{n}\right)u \leq \left(\frac{\lfloor n\sqrt{2} \rfloor}{n}\right)u \leq v \leq \frac{1}{n}(\lfloor n\sqrt{2} \rfloor + 1)u \leq \left(\sqrt{2} + \frac{1}{n}\right)u \ ,$$

for every positive integer n. It follows that $v = u\sqrt{2}$, so that $f(x) = ux$ for every $x \in S$. It is readily checked that this equation satisfies the conditions for all nonnegative u.

478. Solve the equation

$$\sqrt{2+\sqrt{2+\sqrt{2+x}}} + \sqrt{3}\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2x$$

for $x \geq 0$

Solution. Since $2 - \sqrt{2+\sqrt{2+x}} \geq 0$, we must have $0 \leq x \leq 2$. Therefore, there exists a number $t \in [0, \frac{1}{2}\pi]$ for which $\cos t = \frac{1}{2}x$. Now we have that,

$$\begin{aligned}\sqrt{2+\sqrt{2+\sqrt{2+x}}} &= \sqrt{2+\sqrt{2+\sqrt{2+2\cos t}}} \\ &= \sqrt{2+\sqrt{2+\sqrt{4\cos^2(t/2)}}} = \sqrt{2+\sqrt{2+2\cos(t/2)}} \\ &= \sqrt{2+2\cos(t/4)} = 2\cos(t/8)\ .\end{aligned}$$

Similarly, $\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2\sin(t/8)$. Hence the equation becomes

$$2\cos\frac{t}{8} + 2\sqrt{3}\sin\frac{t}{8} = 4\cos t$$

or

$$\frac{1}{2}\cos\frac{t}{8} + \frac{\sqrt{3}}{2}\sin\frac{t}{8} = \cot t\ .$$

Thus,

$$\cos\left(\frac{\pi}{3} - \frac{t}{8}\right) = \cos t\ .$$

Since the argument of the cosine on the left side lies between 0 and $\pi/3$, we must have that $(\pi/3) - (t/8) = t$, or $t = 8\pi/27$.