PROBLEMS FOR SEPTEMBER

Please send your solution to

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no later than October 31, 2006. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes: The *greatest common divisor* of two integers m, n, denoted by $gcd\ (m,n)$ is the largest positive integer which divides (evenly) both m and n. The *least common multiple* of two integers m, n, denoted by $lcm\ (m,n)$ is the smallest positive integer which is divisible by both m and n.

Let n be a positive integer. It can be written uniquely as a sum of powers of 2, *i.e.* in the form

$$n = \epsilon_k \cdot 2^k + \epsilon_{k-1} \cdot 2^{k-1} + \cdots + \epsilon_1 \cdot 2 + \epsilon_0$$

where each ϵ_i takes one of the values 0 and 1. This is known as the *binary representation* of n and is denoted $(\epsilon_k, \epsilon_{k-1}, \cdots, \epsilon_0)_2$. The numbers ϵ_i are known as the *(binary) digits* of n.

The *circumcircle* of a triangle is the centre of the circle that passes through the three vertices of the triangle; the *incentre* of a triangle is centre of the circle within the triangle that is tangent to the three sides; the *orthocentre* of a triangle is the intersection point of its three altitudes.

451. Let a and b be positive integers and let $u = a + b$ and $v = lcm\ (a, b)$. Prove that

$$gcd\ (u, v) = gcd\ (a, b)\ .$$

452. (a) Let m be a positive integer. Show that there exists a positive integer k for which the set

$$\{k + 1, k + 2, \ldots, 2k\}$$

contains exactly m numbers whose binary representation has exactly three digits equal to 1.

(b) Determine all intgers m for which there is exactly one such integer k.

453. Let A, B be two points on a circle, and let AP and BQ be two rays of equal length that are tangent to the circle that are directed counterclockwise from their tangency points. Prove that the line AB intersects the segment PQ at its midpoint.

454. Let ABC be a non-isosceles triangle with circumcentre O, incentre I and orthocentre H. Prove that the angle OIH exceeds $90°$.

455. Let $ABCDE$ be a pentagon for which the position of the base AB and the lengths of the five sides are fixed. Find the locus of the point D for all such pentagons for which the angles at C and E are equal.

456. Let $n + 1$ cups, labelled in order with the numbers $0, 1, 2, \cdots, n$, be given. Suppose that $n + 1$ tokens, one bearing each of the numbers $0, 1, 2, \cdots, n$ are distributed randomly into the cups, so that each cup contains exactly one token.

We perform a sequence of moves. At each move, determine the smallest number k for which the cup with label k has a token with label m not equal to k. Necessarily, $k < m$. Remove this token; move all the tokens in cups labelled $k+1, k+2, \cdots, m$ to the respective cups labelled $k, k+1, m-1$; drop the token with label m into the cup with label m. Repeat.

Prove that the process terminates with each token in its own cup (token k in cup k for each k) in not more that $2^n - 1$ moves. Determine when it takes exactly $2^n - 1$ moves.

457. Suppose that $u_1 > u_2 > u_3 > \cdots$ and that there are infinitely many indices n for which $u_n \geq 1/n$. Prove that there exists a positive integer N for which

$$u_1 + u_2 + u_3 + \cdots + u_N > 2006 \ .$$