Solutions to the September problems

402. Let the sequences $\{x_n\}$ and $\{y_n\}$ be defined, for $n \geq 1$, by $x_1 = x_2 = 10$, $x_{n+2} = x_{n+1}(x_n + 1) + 1$ $(n \geq 1)$ and $y_1 = y_2 = -10$, $y_{n+2} = y_{n+1}(y_n + 1) + 1$ $(n \geq 1)$. Prove that there is no number that is a term of both sequences.

Solution 1. We prove by induction that $x_n \equiv 10$ and $y_n \equiv -10 \equiv 91$ modulo 101 for all positive integers n. The proof is by induction.

The congruences are true for $n = 1, 2$. Suppose that they are true for all natural numbers less than $k + 2$, and in particular that $x_k \equiv x_{k+1} \equiv 10$. Then

$$x_{k+2} \equiv 10(10 + 1) + 1 = 111 \equiv 10$$

modulo 101. Similarly, $y_k \equiv y_{k+1} \equiv -10$ implies

$$y_{k+2} \equiv (-10)(-10 + 1) + 1 = 91 \equiv -10$$

modulo 101. Since x_n and y_n are in different congrence classes modulo 101, the desired result follows.

Comment. How would one hit on this idea? One might get at it by working out the first few terms and seeing that the congruence modulo 101 remains stable. Of one might ask if there is a modulus for which the terms remain in the same modular class as the first two terms. This would require that the modulus m would satisfy $10 \equiv 10(10 + 1) + 1 = 111$ modulo m, whence m should be 101.

Solution 2. [Y. Zhao] Working out the first few terms yields

$$\{x_n\} = \{10, 10, 111, 1222, 136865, 167385896, \cdots\}$$

and

$$\{y_n\} = \{-10, -10, 91, -818, -75255, 61483338, \cdots\} \ .$$

We prove by induction that, for all $n > 2$, (1) $x_n > |y_n|$ and, for all $n > 1$, (2) $|y_{n+1}| > x_n + 2$. In the proof, we repeatedly appeal to $|xy| = |x||y|$ and $|x + 1| \geq |x| - 1$ for all real x and y. (Establish these.)

Ad (1): This holds for $n = 3$ and $n = 4$. Suppose (1) is true for $n < k$. Then

$$\begin{aligned} x_k &= x_{k-1}(x_{k-2} + 1) + 1 > |y_{k-1}|(|y_{k-2}| + 1) + 1 \\ &\geq |y_{k-1}| \cdot |y_{k-2} + 1| + 1 = |y_{k-1}(y_{k-2} + 1)| + 1 \\ &\geq |y_{k-1}(y_{k-2} + 1) + 1| = |y_k| \ . \end{aligned}$$

Thus, (1) follows by an induction argument.

Ad (2): This holds for $n = 2$ and $n = 3$. Suppose (2) is true for $n < k$. Then

$$\begin{aligned} |y_{k+1}| &= |y_k(y_{k-1} + 1) + 1| \geq |y_k(y_{k-1} + 1)| - 1 \\ &= |y_k||y_{k-1} + 1| - 1 \geq |y_k| \cdot (|y_{k-1}| - 1) - 1 > (x_{k-1} + 2)(x_{k-2} + 2 - 1) - 1 \\ &= x_{k-1} \cdot x_{k-2} + 2x_{k-2} + x_{k-1} + 1 > x_{k-1}(x_{k-2} + 1) + 1 + 2 = x_k + 2 \ . \end{aligned}$$

Thus, (2) follows by an induction argument.

Now, we have that $x_2 < |y_3| < x_3 < |y_4| < x_4 < |y_5| < \cdots$, so that no numbers apart from 10 and -10 can appear twice among the terms of the two sequences. The result follows.

403. Let $f(x) = |1 - 2x| - 3|x + 1|$ for real values of x.

(a) Determine all values of the real parameter a for which the equation $f(x) = a$ has two different roots u and v that satisfy $2 \leq |u - v| \leq 10$.

(b) Solve the equation $f(x) = \lfloor x/2 \rfloor$.

Solution 1. [D. Rhee] (a) We have that

$$f(x) = \begin{cases} x+4, & \text{if } x \le -1; \\ -5x-2, & \text{if } -1 < x < \frac{1}{2}; \\ -x-4, & \text{if } x \ge \frac{1}{2}. \end{cases}$$

The graph of $f(x)$ consists of three line segments with nodes at $(-1, 3)$ and $(1/2, -9/2)$. The equation $f(x) = a$ has

• no solutions if $a > 3$;

• two solutions determined by the intersection points of the line $y = a$ and the lines $y = x + 4$ and $y = -5x - 2$ when $-9/2 \le a \le 3$;

• two solutions determined by the intersection points of the line $y = a$ and the lines $y = x + 4$ and $y = -x - 4$ when $a < -9/2$.

When $-9/2 \le a \le 3$, the intersection points in question are $(a-4, a)$ and $(-\frac{1}{5}(a+2), a)$. The inequality $2 \le |u - v| \le 10$ is equivalent to

$$2 \le -\frac{1}{5}(a+2) - (a-4) = \frac{1}{5}(18 - 6a) \le 10$$

(subject to the constraint on a). The values of a that satisfy the requirements of the problem are $-9/2 \le a \le 4/3$.

When $a < -9/2$, the intersection points of the graphs are $(a - 4, a)$ and $(-a - 4, a)$ and the inequality $2 \le |u - v| \le 10$ is equivalent to $2 \le -2a \le 10$. The values of a that satisfy the requirements of the problem are given by $-5 \le a \le -9/2$.

To sum up, the equation $f(x) = a$ has two different solutions u, v that satisfy $2 \le u - v \le 10$ if and only if $-5 \le a \le 4/3$.

(b) Consider the graphs of the functions $f(x)$ and $g(x) \equiv \lfloor x/2 \rfloor$, exploring the possible intersection points of the second graph with the three "branches" of the first graph.

Where do the graphs of $y = g(x)$ and $y = x + 4$ intersect? When $x < -10$, $\lfloor x/2 \rfloor > x/2 - 1 = x/2 - 5 + 1 \ge x/2 + x/2 + 4 = x + 4$, while, when $x > -8$, $\lfloor x/2 \rfloor < x/2 = x/2 + 4 - 4 \le x/2 + 4 + x/2 = x + 4$. Thus, the solution must satisfy $-10 \le x \le 8$, when $\lfloor x/2 \rfloor$ takes one of the two values -5 and -4, We find that $x = -8$ or $x = -9$.

As for the intersection of the graphs of $y = g(x)$ and $y = -5x - 2$, since we are considering only those values of x for which $-1 \le x \le 1/2$ and $g(x)$ is equal to -1 or 0, we find that $x = -1/5$. Finally, when $x \ge 1/2$, $g(x)$ is positive while $-x - 4$ is negative.

Therefore, the only solutions of $f(x) = \lfloor x/2 \rfloor$ are $x = -9, -8, -1/5$.

Solution 2. [Y. Zhao] (a) Similar to the foregoing.

(b) Let $x/2 = n + r$, where n is the integer $\lfloor x/2 \rfloor$ and $0 \le r < 1$. When $x < -1$, we find that $r = -(n + 4)/2$. Since $0 \le r < 1$, $n = -4$ or $n = -5$ so that $x = -8$ or $x = -9$. These check out.

When $-1 \le x \le 1/2$, $r = (-11n - 2)/10$ which leads to $-2/11 \ge n \ge -12/11$ and so $n = -1$ and $r = 9/10$. Thus, $x = -1/5$, which checks out.

When $x > 1/2$, there are no solutions (as in the first solution). The only solutions are $x = -9, -8, -1/5$.

Solution 3. [V. Krakovna] (a) Similar to the foregoing.

(b) This solution is based on a lot of information and relationships from the graphs of $f(x)$ and $g(x)$ and illustrates the geometric approach to solving equations. The equation $f(x) = \lfloor x/2 \rfloor$ has solutions when the line $y = x/2$ comes close (within vertical distance 1) of the graph of $y = f(x)$. For any solution, $f(x)$ must take integer values, as $\lfloor x/2 \rfloor$ always does.

Since $f(x)$ is negative and $x/2$ positive when $x \geq 0$, there are no solutions in this domain. When $-1 \leq x \leq 0$, $x/2$ is negative and the only negative integer values assumed by $f(x)$ are -1 and -2, Only the former yields a solution: $x = -1/5$. When $x < -1$, $x/2 - 1 \leq f(x) \leq x/2$ only when $-10 \leq x \leq -8$, and so $f(x)$ must assume one of the values $-6, -5, -4$. Checking out, we get the solutions $x = -9, -8$.

404. Several points in the plane are said to be *in general position* if no three are collinear.

(a) Prove that, given 5 points in general position, there are always four of them that are vertices of a convex quadrilateral.

(b) Prove that, given 400 points in general position, there are at least 80 nonintersecting convex quadrilaterals, whose vertices are chosen from the given points. (Two quadrilaterals are nonintersecting if they do not have a common point, either in the interior or on the perimeter.)

(c) Prove that, given 20 points in general position, there are at least 969 convex quadrilaterals whose vertices are chosen from these points. (**Bonus:** Derive a formula for the number of these quadrilaterals given n points in general position.)

Solution. [V. Krakovna] (a) Among the five points, there are three X, Y, Z that form a triangle with the other two points M, N outside of the triangle. To see this, select one of the $\binom{5}{3}$ possible triangles that has the smallest area. We note that a convex quadrilateral is characterized by the fact that any two adjacent vertices lie in the same halfplane determined by the side that does not contain these vertices.

Case (i): One of the points M lies outside the triangle on the opposite side of YZ to X but within the angular sector formed by, say, XY and XZ produced. Then $MYXZ$ is a convex quadrilateral.

Case (ii): Both points M and N lie on the opposite side of YZ to X, with M lying in the angle formed by XY and ZY produced and N lying in the angle formed by XZ and YZ produced. Then $MNZY$ is a convex quadrilateral.

Case (iii): Both points M and N lie on the opposite side of YZ to X and in the angle formed by XY and ZY produced. If N and X are on the same side of the line MY, then $XNMY$ is a convex quadrilateral; otherwise, N and Z are on the same side of MY and $MNZY$ is a convex quadrilateral.

(b) Partition the 400 points into groups of five as follows. Choose a line that is not parallel to any of the lines determined by two of the points. Imagine this line passing over the 400 points; it passes over them one at a time, so we can use it to separate off points five at a time. Thus, we have a succession of sets of five points, each included between a pair of lines parallel to the chosen line. From (a), for each group of five, we can select four that are vertices of a convex quadrilateral, so that there are at least 80 convex quadrilaterals determined.

(c) For n points in general position, there are at least $\binom{n}{5}/(n-4)$ convex quadrilaterals. There are $\binom{n}{5}$ possible sets of five points, each of which has a subset of 4 points yielding a convex quadrilateral. However, for each quadrilateral, there are $n - 4$ posibilities for the fifth point of the group to which it might belong, so that each convex quadrilateral could be counted up to $n - 4$ times. Hence, the number of distinct convex quadrilaterals is at least $\binom{n}{5}/(n-4)$. When $n = 20$, this number is 969.

405. Suppose that a permutation of the numbers from 1 to 100, inclusive, is given. Consider the sums of all triples of consecutive numbers in the permutation. At most how many of these sums can be odd?

Solution. [V. Krakovna] In the sequence of 100 numbers, there are 98 triples of consecutive numbers. We will show that it is impossible for all of them to have odd sums.

Let e and o denote an even and an odd number, respectively, in the sequence. A triple has an odd sum if and only if it is (o, o, o) or (e, e, o) in any order. We can get a succession of odd sums by extended sequences of the form $\cdots eeoeeoeeo \cdots$ or $\cdots oooooo \cdots$. However, because there are 50 even numbers, we cannot maintain either of these patterns for the whole sequence. So there must be some transition point between the two types; at such a point, there must be a double e followed or preceded by a string of os, and at least one even triple sum must occur.

However, we can arrange the numbers so that 97 of the 98 triples have odd sums:

$$(2, 4, 1, 6, 8, 3, 10, 12, 5, \cdots, 98, 100, 49, 51, 53, 55, \cdots, 97, 99) ,$$

the only even sum being $100 + 49 + 51$.

406. Let a, b. c be natural numbers such that the expression

$$\frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a}$$

is also equal to a natural number. Prove that the greatest common divisor of a, b and c, $\gcd(a, b, c)$, does not exceed $\sqrt[3]{ab + bc + ca}$, *i.e.*,

$$\gcd(a, b, c) \leq \sqrt[3]{ab + bc + ca} .$$

Solution. [A. Guo] Let k be the sum in the problem and let d be the greatest common divisor of a, b, c. Then $a = du$, $b = dv$, $c = dw$ for some positive integers u, v, w and

$$\begin{aligned} k &= \frac{a+1}{b} + \frac{b+1}{c} + \frac{c+1}{a} \\ &= \frac{a^2c + ac + b^2a + ab + bc + c^2b}{abc} \end{aligned}$$

if and only if

$$ab + bc + ca = kabc - (a^2c + b^2a + c^2b)$$

if and only if

$$d^2(uv + vw + wu) = d^3(kuv - (u^2w + v^2u + w^2v)) .$$

Hence d must divide $uv + vw + wu$, and so d^3 divides, and thus does not exceed, $ab + bc + ca$. The result follows.

407. Is there a pair of natural numbers, x and y, for which

(a) $x^3 + y^4 = 2^{2003}$?

(b) $x^3 + y^4 = 2^{2005}$?

Provide reasoning for your answers to (a) and (b).

Solution. (a) The answer is "no". Consider the equation modulo 13. By Fermat's Little Theorem, $2^{12} \equiv 1$, whence $2^{2003} \equiv 2^{11} = 2048 \equiv 7$ modulo 13. The only possibilities for the congruence of x^3 modulo 13 are 0, 1, 5, 8, 12, and the only possibilities for the congrence of y^4 modulo 13 are 0, 1, 3, 9. We cannot achieve a sum of 7 modulo 13.

(b) The answer is "yes". Observe that

$$2^{2005} = 2^{2004} + 2^{2004} = (2^{668})^3 + (2^{501})^4$$

so that one possible choice is $(x, y) = (2^{668}, 2^{501})$.

408. Prove that a number of the form $a000\cdots0009$ (with $n + 2$ digits for which the first digit a is followed by n zeros and the units digit is 9) cannot be the square of another integer.

Solution. (D. Rhee) It is easy to check that $n \neq 0$. Suppose, if possible, that $n \geq 1$ and that $a \cdot 10^{n+1} + 9 = b^2$. Then $a \cdot 10^{n+1} = (b-3)(b+3)$. The right side is even and the product of two numbers differing by 6; these two numbers have different remainders when divided by 5. Since the left side is a multiple of 5^{n+1}, exactly one of $b-3$ and $b+3$ is divisible by 5^{n+1}. This implies that $b+3$. the greater of the two factors is at least equal to $2 \cdot 5^{n+1}$, while the smaller one is at most equal to $a \cdot 2^n$. Therefore,

$$6 = (b+3) - (b-3) \geq 2 \cdot 5^{n+1} - a \cdot 2^n \geq 2 \cdot 5^{n+1} - 9 \cdot 2^n = 2^n(5^{n+1}/2^{n-1} - 9) \geq 2(25-9) = 32 > 6 \ ,$$

which is false. Hence, our assumption was wrong and there is no integer b for which the given number is equal to its square.