Solutions to the February problems.

348. (b) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x)-3x$ and $f(x)-x^3$ are increasing functions. Must $f(x)-x-x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

Solution 1. Let $u \geq v$. Suppose that $u+v \leq 2$. Then, since $f(x)-3x$ is increasing, $f(u)-3u \geq f(v)-3v$, whence

$$f(u)-f(v) \geq 3(u-v) \geq (u+v+1)(u-v) = u^2-v^2+u-v \Longrightarrow f(u)-u-u^2 \geq f(v)-v-v^2 \ .$$

Suppose that $u+v \geq 2$. Then, since $f(x)-x^3$ is increasing,

$$f(u)-u^3 \geq f(v)-v^3 \Longrightarrow f(u)-f(v) \geq u^3-v^3 = (u-v)(u^2+uv+v^2) \ .$$

Now

$$2[(u^2+uv+v^2)-(u+v+1)] = (u+v)^2+(u-1)^2+(v-1)^2-4 \geq 0 \ ,$$

so that $u^2+uv+v^2 \geq u+v+1$ and

$$f(u)-f(v) \geq (u-v)(u+v+1) = u^2-v^2+u-v \Longrightarrow f(u)-u-u^2 \geq f(v)-v-v^2 \ .$$

Hence $f(u)-u-u^2 \geq f(v)-v-v^2$ whenever $u \geq v$, so that $f(x)-x-x^2$ is increasing. ♠

Solution 2. [F. Barekat] Let $u \geq v$. Then, as in Solution 1, we find that $f(u)-f(v) \geq 3(u-v)$ and $f(u)-f(v) \geq u^3-v^3 = (u-v)(u^2+uv+v^2)$. If $1 \geq u \geq v$, then $3 \geq u+v+1$, so that

$$f(u)-f(v) \geq 3(u-v) \geq (u+v+1)(u-v) \ .$$

If $u \geq v \geq 1$, then $u^2 \geq u$, $v^2 \geq v$ and

$$f(u)-f(v) \geq (u-v)(u^2+uv+v^2) \geq (u-v)(u+1+v) \ .$$

In either case, we have that $f(u)-u-u^2 \geq f(v)-v-v^2$. Finally, if $u \geq 1 \geq v$, then

$$f(u)-u-u^2 \geq f(1)-2 \geq f(v)-v-v^2 \ .$$

The result follows. ♠

Comment. D. Dziabenko assumed that f was differentiable on $\mathbf{R}$, so that $f'(x) \geq 3$ and $f'(x) \geq 3x^2$ everywhere. Hence, for all x, $3f'(x) \geq 3x^2+6$, so that $f'(x) \geq x^2+2 \geq 2x+1$. Hence, the derivative of $f(x)-x-x^2$ is always nonnegative, so that $f(x)-x-x^2$ is increasing. However, there is nothing in the hypothesis that forces f to be differentiable, so this is only a partial solution and its solver would have to settle for a grade of 2 out of 7. A little knowledge is a dangerous thing. If calculus is used, you need to make sure that everything is in place, all assumptions made identified and justified. Often, a more efficient and transparent solution exists without recourse to calculus.

360. Eliminate θ from the two equations

$$x = \cot\theta + \tan\theta$$

$$y = \sec\theta - \cos\theta \ ,$$

to get a polynomial equation satisfied by x and y.

Solution 1. We have that

$$x = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta\cos\theta} \Longrightarrow \sin\theta\cos\theta = \frac{1}{x} \ .$$

$$y = \frac{1}{\cos\theta} - \cos\theta = \frac{\sin^2\theta}{\cos\theta} \ .$$

Hence

$$\sin^3\theta = \frac{y}{x} \qquad \text{and} \qquad \cos^3\theta = \frac{1}{x^2y}$$

so that

$$\left(\frac{y}{x}\right)^{\frac{2}{3}} + \left(\frac{1}{x^2y}\right)^{\frac{2}{3}} = 1 \Longrightarrow (xy^2)^{\frac{2}{3}} + 1 = (x^2y)^{\frac{2}{3}}$$
$$\Longrightarrow (x^2y)^2 = 1 + (xy^2)^2 + 3(xy^2)^{\frac{2}{3}}(x^2y)^{\frac{2}{3}} = 1 + x^2y^4 + 3x^2y^2 \ .$$

Hence

$$x^4y^2 = 1 + x^2y^4 + 3x^2y^2 \ .♠$$

Solution 2. [D. Dziabenko] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2y)$,

$$\begin{aligned}\frac{y^2}{x^2} + \frac{1}{x^4y^2} &= \sin^6\theta + \cos^6\theta \\ &= (\sin^2\theta + \cos^2\theta)^3 - 3(\sin^2\theta + \cos^2\theta)\sin^2\theta\cos^2\theta \\ &= 1 - \frac{3}{x^2} \ .\end{aligned}$$

Hence $x^2y^4 + 1 = x^4y^2 - 3x^2y^2$. ♠

Solution 3. [P. Shi] Using the fact that $\cos^3\theta = 1/(x^2y)$ in the expression for y, we find that

$$y^3 = x^2y - 3y - \frac{1}{x^2y} \Longrightarrow x^2y^4 = x^4y^2 - 3x^2y^2 - 1 \ .♠$$

Solution 4. [Y. Zhao] Since $\sin^3\theta = y/x$ and $\cos^3\theta = 1/(x^2y)$, we have that

$$\sqrt[3]{\frac{y^2}{x^2}} + \sqrt[3]{\frac{1}{x^4y^2}} - 1 = 0 \ .$$

Using the identity $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ with the substitution $a = \sqrt[3]{y^2/x^2}$, $b = \sqrt[3]{1/(x^4y^2)}$, $c = -1$, we obtain that

$$\frac{y^2}{x^2} + \frac{1}{x^4y^2} - 1 + \frac{3}{x^2} = 0$$

or

$$x^2y^4 + 1 - x^4y^2 + 3x^2y^2 = 0 \ .♠$$

Comment. In a question like this, it is very easy to make a mechanical slip. Accordingly, it is prudent to make a convenient substitution of values to see if your identity works. For example, when $\theta = \pi/4$, $x = 2$ and $y = 1/\sqrt{2}$, and we find that the identity checks out.

361. Let $ABCD$ be a square, M a point on the side BC, and N a point on the side CD for which $BM = CN$. Suppose that AM and AN intersect BD and P and Q respectively. Prove that a triangle can be constructed with sides of length $|BP|$, $|PQ|$, $|QD|$, one of whose angles is equal to $60°$.

Solution 1. Let the sides of the square have length 1 and let $|BM| = u$. Then $|NC| = u$ and $|MC| = |ND| = 1 - u$. Let $|BP| = a$, $|PQ| = b$ and $|QD| = c$. Since triangles APD and MPB are similar, $(a/u) = b + c$. Since triangle DQN and BQA are similar, $(c/(1-u))) = a + b$. Hence

$$(1 - u + u^2)a = (2u - u^2)b \qquad \text{and} \qquad (1 - u + u^2)c = (1 - u^2)b$$

so that

$$a : b : c = (2u - u^2) : (1 - u + u^2) : (1 - u^2) \ .$$

Now

$$\begin{aligned}(2u - u^2)^2 + (1 - u^2)^2 - 2(2u - u^2)(1 - u^2)\cos 60^\circ \\ &= (4u^2 - 4u^3 + u^4) + (1 - 2u^2 + u^4) - (2u - 2u^3 - u^2 + u^4) \\ &= u^4 - 2u^3 + 3u^2 - 2u + 1 = (1 - u + u^2)^2 \ .\end{aligned}$$

Thus $b^2 = a^2 + c^2 - ac$. Note that this implies that $(a - c)^2 < b^2 < (a + c)^2$, whence $a < b + c$, $b < a + c$ and $c < a + b$. Accordingly, a, b, c are the sides of a triangle and $b^2 = a^2 + c^2 = \frac{1}{2}\cos 60^\circ$. From the law of cosines, the result follows. ♠

Solution 2. [F. Barekat] *Lemma.* Let PQR be a right triangle with $\angle R = 90^\circ$, $|QR| = p$ and $|PR| = q$. Then the length m of the bisector RT of angle R (with T on PQ) is equal to $(\sqrt{2}pq)/(p + q)$.

Proof. $[PQR] = [QRT] + [PRT] \Longrightarrow \frac{1}{2}pq = \frac{1}{2}pm\sin 45^\circ + \frac{1}{2}qm\sin 45^\circ$, from which the result follows. ♣

Using the same notation as in Solution 1, the above lemma and the fact that BD bisects the right angles at B and D, we find that

$$a = \frac{\sqrt{2}u}{1 + u} \ , \quad c = \frac{\sqrt{2}(1 - u)}{2 - u} \ , \quad b = \sqrt{2} - a - c \ .$$

Hence

$$\begin{aligned}(a^2 + c^2 - 2ac\cos 60^\circ) - b^2 &= (a^2 + c^2 - ac) - (2 + a^2 + c^2) + 2\sqrt{2}(a + c) - 2ac \\ &= 2\sqrt{2}(a + c) - 3ac - 2 \\ &= \frac{2}{(1 + u)(2 - u)}[2u(2 - u) + 2(1 + u)(1 - u) - 3u(1 - u) - (1 + u)(2 - u)] \\ &= \frac{2}{(1 + u)(2 - u)}[4u - 2u^2 + 2 - 2u^2 - 3u + 3u^2 - 2 - u + u^2] = 0 \ .\end{aligned}$$

Hence $b^2 = a^2 + c^2 - 2ac\cos 60^\circ$ and the result follows from the law of cosines. ♠

Comment. P. Shi used Menelaus' Theorem with triangle BOC and transversal APM and with triangle COD and transversal AQN to get

$$\frac{|BP|}{|PO|} \cdot \frac{|OA|}{|AC|} \cdot \frac{|CM|}{|MB|} = \frac{|DQ|}{|QO|} \cdot \frac{|OA|}{|AC|} \cdot \frac{|CN|}{|ND|} = 1 \ ,$$

where O is the centre of the square. Noting that $|PO| = (1/\sqrt{2}) - |BP|$ and $|QO| = (1/\sqrt{2}) - |DQ|$, we can determine the lengths of BP and DQ.

362. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B, C meet the circle again at U, V, W, respectively. Prove that the area of triangle UVW is not less than the area of triangle ABC.

Solution 1. Let R be the common circumradius of the triangles ABC and UVW. Observe that $\angle WUA = \angle WCA = \frac{1}{2}\angle ACB$ and $\angle VUA = \angle VBA = \frac{1}{2}\angle ABC$, whence

$$U = \angle WUV = \frac{1}{2}(\angle ACB + \angle ABC) = \frac{1}{2}(B + C) \ ,$$

et cetera. Now

$$[ABC] = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C$$

and

$$[UVW] = \frac{uvw}{4R} = 2R^2 \sin U \sin V \sin W \ .$$

Since by the arithmetic-geometric means inequality,

$$\begin{aligned}\sqrt{\sin A \sin B} &= 2\sqrt{(\sin\frac{A}{2}\cos\frac{B}{2})(\cos\frac{A}{2}\sin\frac{B}{2})} \\ &\le \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2} \\ &= \sin\frac{A+B}{2} = \sin W \ ,\end{aligned}$$

et cetera, it follows that $[ABC] \le [UVW]$ with equality if and only if ABC is an equilateral triangle. ♠

Second solution. Since $\frac{1}{2}(A+B) = 90^\circ - \frac{C}{2}$, we find that

$$[ABC] = 2R^2 \sin A \sin B \sin C = 16R^2 \sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}$$

and

$$[UVW] = 2R^2 \sin U \sin V \sin W = 2R^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \ ,$$

so that

$$\begin{aligned}\frac{[ABC]}{[UVW]} &= 8 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 8\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= \frac{8(s-a)(s-b)(s-c)}{abc} = \frac{8[ABC]^2}{sabc} = \frac{2r}{R} \le 1\end{aligned}$$

bu Euler's inequality for the inradius and the circumradius. The result follows. ♠

Third solution. Let I be the incentre and H the orthocentre of triangle ABC. Suppose the respective altitudes from A, B, C meet the circumcircle at P, Q, R. We have that

$$\angle BHP = \angle AHQ = 90^\circ - \angle HAC = 90^\circ - \angle PAC = 90^\circ - \angle PBC = \angle BPH$$

so that $BH = BP$. Similarly, $CH = CP$. Hence $\Delta HBC \equiv \Delta PBC$ (SSS). Similarly $\Delta HAC \equiv \Delta QAC$ and $\Delta HAB \equiv \Delta RAB$, whence $[ARBPCQ] = 2[ABC]$.

Let AU intersect VW at X. Then

$$\begin{aligned}\angle VXA &= \angle XWA + \angle XAW = \angle VWA + \angle UAB + \angle WAB \\ &= \angle VBA + \angle UAB + \angle WCB = \frac{1}{2}(\angle B + \angle A + \angle C) = 90^\circ \ ,\end{aligned}$$

so that UA is an altitude of triangle UVW, as similarly are VB and WC (so that I is the orthocentre of triangle UVW). Therefore, we have, as above, that $[UCVAWB] = 2[UVW]$.

But U is the midpoint of the arc BC, V the midpoint of the arc CA and W the modpoint of the orc AB. Thus, $[CPB] \le [CUB]$, $[AQC] \le [AVC]$ and $[BRA] \le [BWA]$. Therefore, $[ARBPCQ] \le [UCVAWB]$ and so $[ABC] \le [UVW]$. ♠

Comment. F. Barekat noted that for $0 < x < \pi$, the function $\log \sin x$ is concave so that $\sqrt{\sin u \sin v} \le \sin(\frac{1}{2}(u+v))$ for $0 \le u, v \le \pi$. (This can be seen by noting that the second derivative of $\log \sin x$ is $-\csc^2 x < 0$.) Then the solution can be completed as in Solution 1.

363. Suppose that x and y are positive real numbers. Find all real solutions of the equation

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \sqrt{xy} + \frac{x+y}{2} \ .$$

Preliminaries. It is clear that if one of x and y vanishes, then so must the other. Otherwise, there are two possibilities, according as x and y are both negative or both positive ($\sqrt{xy}$ needs to make sense). If x and y are both negative, then the only solution is $x = y$ as the equation asserts that the sum of the harmonic and geometric means of $-x$ and $-y$ is equal to the sum of the arithmetic mean and root-mean-square of these quantities. For unequal positive reals, each of the first two is less than each of the second two. Henceforth, we assume that x and y are positive. Let $h = 2xy/(x+y)$, $g = \sqrt{xy}$, $a = \frac{1}{2}(x+y)$ and $r = \sqrt{\frac{1}{2}(x^2+y^2)}$.

Solution 1. It is straightforward to check that $2a^2 = r^2 + g^2$ and that $g^2 = ah$. Suppose that $h+r = a+g$. Then

$$\begin{aligned} r &= a + g - h \\ &\Longrightarrow 2a^2 - g^2 = r^2 = a^2 + g^2 + h^2 - 2ah - 2gh + 2ag \\ &\Longrightarrow (a+h)(a-h) = a^2 - h^2 = 2(g^2 - ah) + 2g(a-h) = 0 + 2g(a-h) \\ &\Longrightarrow a = h \quad \text{or} \quad a + h = 2g \ . \end{aligned}$$

In the latter case, $g = \frac{1}{2}(a+h) = \sqrt{ah}$, so that both possibilities entail $a = g = h$. But equality of these means occur if and only if $x = y$. ♠

Solution 2. [A. Cornuneanu] Observe that

$$r^2 - g^2 = \frac{1}{2}(x-y)^2 = 2a(a-h) \ .$$

Since, from the given equation,

$$r^2 - g^2 = (r-g)(r+g) = (a-h)(r+g) \ ,$$

it follows that $a = \frac{1}{2}(r+g)$. However, it can be checked that, in general,

$$a = \sqrt{\frac{r^2+g^2}{2}} \ ,$$

so that a is at once the arithmetic mean and the root-mean-square of r and g. But this can occur if and only if $r = g = a$ if and only if $x = y$. ♠

Solution 3. [R. O'Donnell] From the homogeneity of the given equation, we can assume without loss of generality that $g = 1$ so that $y = 1/x$. Then $h = 1/a$ and $r = \sqrt{2a^2 - 1}$. The equation becomes

$$r + \frac{1}{a} = a + 1 \qquad \text{or} \qquad \sqrt{2a^2-1} = 1 + a - \frac{1}{a} \ .$$

Squaring and manipulating leads to

$$0 = a^4 - 2a^3 + 2a - 1 = (a-1)^3(a+1)$$

whence $a = 1 = g$ and so $x = y = 1$. The main result follows from this. ♠

Solution 4. [D. Dziabenko] Let $2a = x + y$ and $2b = x - y$, so that $x = a + b$ and $y = a - b$, Then $a \neq 0$, $a \geq b$ and the equation becomes

$$\frac{a^2-b^2}{a} + \sqrt{a^2+b^2} = \sqrt{a^2-b^2} + a \Longleftrightarrow \sqrt{a^2+b^2} - \sqrt{a^2-b^2} = \frac{b^2}{a} \ .$$

Multiplying by $\sqrt{a^2+b^2}+\sqrt{a^2-b^2}$ yields that

$$2b^2 = (a^2+b^2)-(a^2-b^2) = \frac{b^2}{a}(\sqrt{a^2+b^2}+\sqrt{a^2-b^2}) ,$$

from which

$$\frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{2} = a = \sqrt{\frac{(a^2+b^2)+(a^2-b^2)}{2}} .$$

The left side is the arithmetic mean and the right the root-mean-square of $\sqrt{a^2+b^2}$ and $\sqrt{a^2-b^2}$. These are equal if and only if $a^2+b^2 = a^2-b^2 \Leftrightarrow b = 0 \leftrightarrow x = y$. ♠

364. Determine necessary and sufficient conditions on the positive integers a and b such that the vulgar fraction a/b has the following property: *Suppose that one successively tosses a coin and finds at one time, the fraction of heads is less than a/b and that at a later time, the fraction of heads is greater than a/b; then at some intermediate time, the fraction of heads must be exactly a/b.*

Solution. Consider the situation in which a tail is tossed first, and then a head is tossed thereafter. Then the fraction of heads after n tosses is $(n-1)/n$. Since any positive fraction a/b less than 1 exceeds this for $n=1$ and is less than this for n sufficiently large, a/b can be realized as a fraction of head tosses only if it is of this form (*i.e.* $a = b-1$).

On the other hand, suppose that $a/b = (n-1)/n$ for some positive integer n. There must exist numbers p and q for which the fraction p/q of heads at one toss is less than a/b and the fraction $(p+1)/(q+1)$ at the next toss is not less than a/b. Thus

$$\frac{p}{q} < \frac{n-1}{n} \leq \frac{p+1}{q+1} .$$

Hence $np < nq - q$ and $nq - q + n - 1 \leq np + n$, so that

$$np < nq - q \leq np + 1 .$$

Since the three members of this inequality are integers and the outer two are consecutive, we must have $nq - q = np + 1$, whence

$$\frac{n-1}{n} = \frac{p+1}{q+1} .$$

Hence the necessary and sufficient condition is that $a/b = (n-1)/n$ for some positive integer n. ♠

Rider. What is the situation when the fraction of heads moves from a number greater than a/b to a number less than a/b?

365. Let $p(z)$ be a polynomial of degree greater than 4 with complex coefficients. Prove that $p(z)$ must have a pair u, v of roots, not necessarily distinct, for which the real parts of both u/v and v/u are positive. Show that this does not necessarily hold for polynomials of degree 4.

Solution. Since the degree of the polynomial exceeds 4, there must be two roots u, v in one of the four quadrants containing a ray from the origin along either the real or the imaginary axis along with all the points within the region bounded by this ray and the next such ray in the counterclockwise direction. The difference in the arguments between two such numbers must be strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\arg(u/v) = \arg u - \arg v$ and $\arg(v/u) = \arg v - \arg u$ both lie in this range, both u/v and v/u lie to the right of the imaginary axis, and so have positive real parts.

This result does not necessarily hold for a polynomial of degree 4, as witnessed by $z^4 - 1$ whose roots are $1, -1, i, -i$.

366. What is the largest real number r for which

$$\frac{x^2+y^2+z^2+xy+yz+zx}{\sqrt{x}+\sqrt{y}+\sqrt{z}} \geq r$$

holds for all positive real values of x, y, z for which $xyz = 1$.

Solution 1. Let u, v, w be positive reals for which $u^2 = yz$, $v^2 = zx$ and $w^2 = xy$. Then $\sqrt{x} = x\sqrt{yz} = xu$, $\sqrt{y} = yv$ and $\sqrt{z} = zw$, so that

$$\begin{aligned}(x^2+y^2+z^2)+(xy+yz+zx) &= (x^2+y^2+z^2)+(u^2+v^2+w^2)\\ &\geq 2\sqrt{x^2+y^2+z^2}\sqrt{u^2+v^2+w^2} \geq 2(xu+yv+zw)\ ,\end{aligned}$$

from the arithmetic-geometric means and the Cauchy-Schwarz inequalities. Hence, the inequality is always valid when $r \leq 2$. When $(x, y, z) = (1, 1, 1)$, the left side is equal to 2, so the inequality does not always hold when $r > 2$. Hence the largest value of r is 2. ♠

Solution 2. App;ying the arithmetic-geometric means inequality to the left side yields

$$\begin{aligned}\frac{(x^2+yz)+(y^2+zx)+(z^2+xy)}{\sqrt{x}+\sqrt{y}+\sqrt{z}} &\geq \frac{2(\sqrt{x^2yz}+\sqrt{y^2zx}+\sqrt{z^2xy}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\\ &= \frac{2\sqrt{xyz}(\sqrt{x}+\sqrt{y}+\sqrt{z})}{\sqrt{x}+\sqrt{y}+\sqrt{z}} = 2\ .\end{aligned}$$

Equality occurs when $(x, y, z) = (1, 1, 1)$. Hence the largest value of r for which the inequality always holds is 2. ♠