Solutions.

297. The point P lies on the side BC of triangle ABC so that $PC = 2BP$, $\angle ABC = 45°$ and $\angle APC = 60°$. Determine $\angle ACB$.

Solution 1. Let D be the image of C under a reflection with axis AP. Then $\angle APC = \angle APD = \angle DPB = 60°$, $PD = PC = 2BP$, so that $\angle DBP = 90°$. Hence AB bisects the angle DBP, and AP bisects the angle DPC, whence A is equidistant from BD, PC and PD.

Thus, AD bisects $\angle EDP$, where E lies on BD produced. Thus

$$\begin{aligned}\angle ACB = \angle ADP &= \frac{1}{2}\angle EDP \\ &= \frac{1}{2}(180° - \angle BDP) = \frac{1}{2}(180° - 30°) = 75° \ .\end{aligned}$$

Solution 2. [Y. Zhao] Let Q be the midpoint of PC and R the intersection of AP and the right bisector of PQ, so that $PR = QR$ and $BR = CR$. Then $\angle RPQ = \angle RQP = 60°$ and triangle PQR is equilateral. Hence $PB = PQ = PR = RQ = QC$ and $\angle PBR = \angle PRB = \angle QRC = \angle QCR = 30°$.

Also, $\angle RBA = 15° = \angle PAB = \angle RAB$, so $AR = BR = CR$. Thus, $\angle RAC = \angle RCA$. Now $\angle ARC = 180° - \angle PRQ - \angle QRC = 90°$, so that $\angle RCA = 45°$ and $\angle ACB = 75°$.

Solution 3. [R. Shapiro] Let H be the foot of the perpendicular from C to AP. Then CPH is a $30-60-90$ triangle, so that $BP = \frac{1}{2}PC = PH$ and $\angle PBH = \angle PHB = 30° = \angle PCH$. Hence, $BH = HC$. As

$$\angle HAB = \angle PAB = 180° - 120° - 45° = 15° = \angle ABP - \angle HBP = \angle ABH \ ,$$

$AH = BH = HC$. Therefore, $\angle HAC = \angle HCA = 45°$. Thus, $\angle ACB = \angle HCA + \angle PCH = 75°$.

Solution 4. From the equation expressing $\tan 30°$ in terms of $\tan 15°$, we find that $\tan 15° = 2 - \sqrt{3}$ and $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$. Let $\angle ACP = \theta$, so that

$$\angle PAC = 180° - 60° - \theta = 120° - \theta \ .$$

Suppose, wolog, we set $|BP| = 1$, so that $|PC| = 2$. Then by the Law of Sines in triangle ABP,

$$|AP| = \frac{\sin 45°}{\sin 15°} = \sqrt{3} + 1.$$

By the Law of Sines in triangle APC,

$$\frac{\sin\theta}{\sqrt{3}+1} = \frac{\sin(120° - \theta)}{2} = \frac{\sqrt{3}\cos\theta}{4} + \frac{\sin\theta}{4}$$

whence $(3 - \sqrt{3})\sin\theta = (3 + \sqrt{3})\cos\theta$. Hence

$$\tan\theta = 2 + \sqrt{3} = (2 - \sqrt{3})^{-1} = (\tan 15°)^{-1} \ ,$$

so that $\theta = 75°$.

298. Let O be a point in the interior of a quadrilateral of area S, and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \ .$$

Prove that $ABCD$ is a square with centre O.

Solution.

$$\begin{aligned}
&|OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \\
&\quad = \frac{1}{2}(|OA|^2 + |OB|^2) + \frac{1}{2}(|OB|^2 + |OC|^2) + \frac{1}{2}(|OC|^2 + |OD|^2) + \frac{1}{2}(|OD|^2 + |OA|^2) \\
&\quad \geq |OA||OB| + |OB||OC| + |OC||OD| + |OD||OA| \\
&\quad \geq 2[AOB] + 2[BOC] + 2[COD] + 2[DOA] = 2S
\end{aligned}$$

with equality if and only if $OA = OB = OC = OD$ and all the angles AOB, BOC, COD and DOA are right. The result follows.

299. Let $\sigma(r)$ denote the sum of all the divisors of r, including r and 1. Prove that there are infinitely many natural numbers n for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever $1 \leq k < n$.

Solution 1. Let $u_m = \sigma(m)/m$ for each positive integer m. Since $d \leftrightarrow 2d$ is a one-one correspondence between the divisors of m and some even divisor of $2m$, $\sigma(2m) \geq 2\sigma(m) + 1$, so that

$$u_{2m} = \frac{\sigma(2m)}{2m} \geq \frac{2\sigma(m) + 1}{2m} > \frac{\sigma(m)}{m} = u_m$$

for each positive integer m.

Let r be a given positive integer, and select $s \leq 2^r$ such that $u_s \geq u_k$ for $1 \leq k \leq 2^r$ (*i.e.*, u_s is the largest value of u_k for k up to and including 2^r). Then, as $u_{2s} > u_s$, it must happen that $2^r \leq 2s \leq 2^{r+1}$ and $u_{2s} \geq u_k$ for $1 \leq k \leq 2^r$.

Suppose that n is the smallest positive integer t for which $2^r \leq t$ and $u_k \leq u_t$ for $1 \leq k \leq 2^r$. Then $2^r \leq n \leq 2s \leq 2^{r+1}$. Suppose that $1 \leq k \leq n$. If $1 \leq k \leq 2^r$, then $u_k \leq u_n$ from the definition of n. If $2^r < k < n$, then there must be some number k' not exceeding 2^r for which $u_k < u_{k'} \leq u_n$. Thus, n has the desired property and $2^r \leq n \leq 2^{r+1}$. Since such n can be found for each positive exponent r, the result follows.

Comment. The sequence selected in this way starts off: $\{1, 2, 4, 6, 12, \cdots\}$.

Solution 2. [P. Shi] Define u_m as in Solution 1. Suppose, if possible, that there are only finitely many numbers n satisfying the condition of the problem. Let N be the largest of these, and let u_s be the largest value of u_m for $1 \leq m \leq N$. We prove by induction that $u_n \leq u_s$ for every positive integer n. This holds for $n \leq N$. Suppose that $n > N$. Then, there exists an integer $r < n$ for which $u_r > u_n$. By the induction hypothesis, $u_r \leq u_s$, so that $u_n < u_s$. But this contradicts the fact (as established in Solution 1) that $u_{2s} > u_s$.

300. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90°$, and let $\mathcal{K}$ be its circumcircle. Suppose that the tangent to $\mathcal{K}$ at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular from A to BE and Y the midpoint of AX. Suppose that BY meets $\mathcal{K}$ again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.

Solution 1. Let AZ and BD intersect at M, and AE and BC intersect at P. Since PY joints the midpoints of two sides of triangle AEX, $PY \| EX$. Since $\angle APY = \angle AEB = \angle AZB = \angle AZY$, the quadrilateral $AZPY$ is concyclic. Since $\angle AYP = \angle AXE = 90°$, AP is a diameter of the circumcircle of $AZPY$ and BD is a tangent to this circle. Hence $MP^2 = MZ \cdot MA$. Since

$$\angle PAD = \angle EAD = \angle EBA = \angle XBA,$$

triangles PAD and XBA are similar. Since

$$\angle MAD = \angle ZAD = \angle ZBA = \angle YBA,$$

it follows that

$$\angle PAM = \angle PAD - \angle MAD = \angle XBA - \angle YBA = \angle XBY$$

so that triangles PAM and XBY are similar. Thus

$$\frac{PM}{AP} = \frac{XY}{XB} = \frac{XA}{2XB} = \frac{PD}{2PA} \Longrightarrow PD = 2PM \Longrightarrow MD = PM \ .$$

Hence $MD^2 = MP^2 = MZ \cdot MA$ and the desired result follows.

Solution 2. [Y. Zhao] As in Solution 1, we see that there is a circle through the vertices of $AZPY$ and that BD is tangent to this circle. Let O be the centre of the circle $\mathcal{K}$. The triangles OPA and OAD are similar, whereupon $OP{\cdot}OD = OA^2$. The inversion in the circle $\mathcal{K}$ interchanges P and D, carries the line BD to itself and takes the circumcircle of triangle AZP to the circumcircle of triangle AZD. As the inversion preserves tangency of circles and lines, the desired result follows.

301. Let $d = 1, 2, 3$. Suppose that M_d consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference d. Prove that, if $c \in M_3$, then there exist integers $a \in M_1$ and $b \in M_2$ for which $c = ab$.

Solution. M_1 consists of all the powers of 2, and M_2 consists of 1 and all the primes. We prove these assertions.

Since $k + (k + 1) = 2k + 1$, every odd integer exceeding 1 is the sum of two consecutive terms. Indeed, for each positive integers m and r,

$$(m - r) + (m - r + 1) + \cdots + (m - 1) + m + (m + 1) + \cdots + (m + r - 1) + (m + r) = (2r + 1)m \ ,$$

and,

$$m + (m + 1) + \cdots + (m + 2r - 1) = r[2(m + r) - 1] \ ,$$

so that it can be deduced that every positive integer with at least one odd positive divisor exceeding 1 is the sum of consecutives, and no power of 2 can be so expressed. (If $m < r$ in the first sum, the negative terms in the sum are cancelled by positive ones.) Thus, M_1 consists solely of all the powers of 2.

Since $2n = (n + 1) + (n - 1)$, M_2 excludes all even numbers exceeding 2. Let $k \geq 2$ and $m \geq 1$. Then

$$m + (m + 2) + \cdots + (m + 2(k - 1)) = km + k(k - 1) = k(m + k - 1)$$

so that M_2 excludes all multiples of k from k^2 on. Since all such numbers are composite, M_2 must include all primes. Since each composite number is at least as large as the square of its smallest nontrivial divisor, each composite number must be excluded from M_2.

We now examine M_3. The result will be established if we show that M_3 does not contain any number of the form $2^r uv$ where r is a nonnegative integer and u, v are odd integers with $u \geq v > 1$. Suppose first that $r \geq 1$ and let $a = 2^r u - \frac{3}{2}(v - 1)$. Then

$$a \geq 2u - \frac{3}{2}(v - 1) \geq \frac{v}{2} + 1 > 1$$

and

$$a + (a + 3) + \cdots + [a + 3(v - 1)] = v[a + (3/2)(v - 1)] = 2^r uv \ .$$

Since $m + (m+3) = 2m+3$, we see that M_3 excludes all odd numbers exceeding 3, and hence all odd composite numbers. Hence, every number in M_3 must be the product of a power of 2 and an odd prime or 1.

Comment. The solution provides more than necessary. It suffices to show only that M_1 contains all powers of 2, M_2 contains all primes and M_3 excludes all numbers with a composite odd divisor.

302. In the following, $ABCD$ is an arbitrary convex quadrilateral. The notation $[\cdots]$ refers to the area.

(a) Prove that $ABCD$ is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD] \ .$$

(b) Suppose that F is an interior point of the quadrilateral $ABCD$ such that $ABCF$ is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD] \ .$$

Solution 1. (a) Suppose that AB is not parallel to CD. Wolog, let these lines meet at E with A between E and B, and D between E and C. Let P, Q, R, S be the respective feet of the perpendiculars from A to CD, B to CD, C to AB, D to AB produced. Then

$$[ABC] \cdot [ACD] = [ABD][BCD] \Leftrightarrow |AB||CR||CD||AP| = |AB||DS||CD||BQ| \Leftrightarrow CR : DS = BQ : AP \ .$$

By similar triangles, we find that $CE : DE = CR : DS = BQ : AP = BE : AE$. The dilation with centre E and factor $|AE|/|BE|$ takes B to A, C to D and so the segment BC to the parallel segment AD. Thus $ABCD$ is a trapezoid.

(b) Let the quadrilateral be in the horizontal plane of three-dimensional space and let F be at the origin of vectors. Suppose that $\mathbf{u} = \overrightarrow{FA}$, $\mathbf{v} = \overrightarrow{FC}$, and $-p\mathbf{u} - q\mathbf{v} = \overrightarrow{FD}$, where p and q are nonnegative scalars. We have that $\overrightarrow{FB} = \mathbf{u} + \mathbf{v}$. Then

$$2[ABC] = |\mathbf{u} \times \mathbf{v}| \ ;$$

$$\begin{aligned} 2[ACD] &= 2([FAC] + [FAD] + [FCD]) \\ &= |\mathbf{u} \times \mathbf{v}| + |\mathbf{u} \times (p\mathbf{u} + q\mathbf{v})| + |\mathbf{v} \times (p\mathbf{u} + q\mathbf{v})| \\ &= (1 + q + p)|\mathbf{u} \times \mathbf{v}| \ ; \end{aligned}$$

$$2[FCD] = p|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[AFD] = q|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[ABD] = |(p\mathbf{u} + q\mathbf{v} + \mathbf{u}) \times \mathbf{v}| = (1+p)|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[BCD] = |(p\mathbf{u} + q\mathbf{v} + \mathbf{v}) \times \mathbf{u}| = (1+q)|\mathbf{u} \times \mathbf{v}|| \ .$$

The result follows.

Solution 2. [Y. Zhao] Observe that, since $(A + C) + (B + D) = 360°$,

$$\begin{aligned} \sin A \sin C - \sin B \sin D &= \frac{1}{2}[\cos(A - C) - \cos(A + C) - \cos(B - D) + \cos(B + D)] \\ &= \frac{1}{2}[\cos(A - C) - \cos(B - D)] = \frac{1}{2}[\cos(B + A - B - C) - \cos(B + A + B + C)] \\ &= \sin(B + A)\sin(B + C) \ . \end{aligned}$$

(a) Hence

$$\begin{aligned} 4[ABD][BCD] - 4[ABC][ACD] &= (AB \cdot DA \sin A)(BC \cdot CD \sin C) - (AB \cdot BC \sin B)(CD \cdot DA \sin D) \\ &= (AB \cdot BC \cdot CD \cdot DA)(\sin A \sin C - \sin B \sin D) \\ &= (AB \cdot BC \cdot CD \cdot DA)\sin(B + A)\sin(B + C) \ . \end{aligned}$$

The left side vanishes if and only if $A+B=C+D=180°$ or $B+C=A+D=180°$, *i.e.*, $AD\|BC$ or $AB\|CD$.

(b) From (a), we have that

$$\begin{aligned}4[ABD][BCD]-4[ABC][ACD] &= (AB\cdot BC\cdot CD\cdot DA)\sin(A+B)\sin(B+C)\\ &= (AB\cdot BC\cdot CD\cdot DA)\sin(A+B-180°)\sin(B+C-180°)\\ &= (FC\cdot AF\cdot CD\cdot DA)(\sin(\angle BAD-\angle BAF)\sin(\angle BCD-\angle BCF))\\ &= [(DA\cdot AF)\sin\angle DAF][(DC\cdot CF)\sin\angle DCF]\\ &= 4[AFD][FCD]\ ,\end{aligned}$$

as desired.

303. Solve the equation

$$\tan^2 2x = 2\tan 2x\tan 3x + 1\ .$$

Solution 1. Let $u=\tan x$ and $v=\tan 2x$. Then

$$v^2-2v\left(\frac{u+v}{1-uv}\right)-1=0$$

$$\iff v^2-uv^3-2uv-2v^2-1+uv=0$$

$$\iff 0=uv+1+v^2+uv^3=(uv+1)(1+v^2)$$

$$\iff uv=-1\ .$$

Now $v=2u(1-u^2)^{-1}$, so that $2u=v-u^2v=u+v$ and $u=v$. But then $u^2=-1$ which is impossible. Hence the equation has no solution.

Solution 2.

$$\begin{aligned}0&=\tan^2 2x-2\tan 3x\tan 2x-1\\ &=\tan^2 2x-2\tan 3x\tan 2x+\tan^2 3x-\sec^2 3x\\ &=(\tan 2x-\tan 3x)^2-\sec^2 3x\\ &=(\tan 2x-\tan 3x-\sec 3x)(\tan 2x-\tan 3x+\sec 3x)\ .\end{aligned}$$

Hence, either $\tan 2x=\tan 3x+\sec 3x$ or $\tan 2x=\tan 3x-\sec 3x$. Suppose that the former holds. Multiplying the equation by $\cos 2x\cos 3x$ yields $\sin 2x\cos 3x=\sin 3x\cos 2x+\cos 2x$. Hence

$$\begin{aligned}0&=\cos 2x+(\sin 3x\cos 2x-\sin 2x\cos 3x)\\ &=1-2\sin^2 x+\sin x=(1-\sin x)(1+2\sin x)\ ,\end{aligned}$$

whence

$$x\equiv\frac{\pi}{2},-\frac{\pi}{6},\frac{7\pi}{6}$$

modulo 2π. But $\tan 3x$ is not defined at any of these angles, so the equation fails. Similarly, in the second case, we obtain $0=(2\sin x-1)(\sin x+1)$ so that

$$x\equiv\frac{-\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$$

modulo 2π, and the equation again fails. Thus, there are no solutions.

Solution 3. Let $t=\tan x$, so that $\tan 2x=2t(1-t^2)^{-1}$ and $\tan 3x=(3t-t^3)(1-3t^2)^{-1}$. Substituting for t in the equation and clearing fractions leads to

$$4t^2(1-3t^2)=4t(3t-t^3)(1-t^2)+(1-t^2)^2(1-3t^2)$$

$$\Leftrightarrow 4t^2 - 12t^4 = (12t^2 - 16t^4 + 4t^6) + (1 - 5t^2 + 7t^4 - 3t^6)$$
$$\Leftrightarrow 0 = t^6 + 3t^4 + 3t^2 + 1 = (t^2+1)^3 \ .$$

There are no real solutions to the equation.

Solution 4. The equation is undefined if $2x$ or $3x$ is an odd multiple of $\pi/2$. We exclude this case. Then the equation is equivalent to

$$\frac{\sin^2 2x - \cos^2 2x}{\cos^2 2x} = \frac{2\sin 2x \sin 3x}{\cos 2x \cos 3x}$$

or

$$\begin{aligned} 0 &= \frac{2\sin 2x \sin 3x}{\cos 2x \cos 3x} + \frac{\cos 4x}{\cos^2 2x} \\ &= \frac{\sin 4x \sin 3x + \cos 4x \cos 3x}{\cos^2 2x \cos 3x} \\ &= \frac{\cos x}{\cos^2 2x \cos 3x} \ . \end{aligned}$$

Since $\cos x$ vanishes only if x is an odd multiple of π, we see that the equation has no solution.

Solution 5. [Y. Zhao] Observe that, when $\tan(A-B) \neq 0$,

$$1 + \tan A \tan B = \frac{\tan A - \tan B}{\tan(A-B)} \ .$$

In particular,

$$1 + \tan x \tan 2x = \frac{\tan 2x - \tan x}{\tan x} \quad \text{and} \quad 1 + \tan 2x \tan 3x = \frac{\tan 3x - \tan 2x}{\tan x} \ .$$

There is no solution when $x \equiv 0 \pmod{\pi}$, so we exclude this possibility. Thus

$$\begin{aligned} 0 &= (1 + \tan 2x \tan 3x) + (\tan 2x \tan 3x - \tan^2 2x) \\ &= (\tan 3x - \tan 2x)(\cot x + \tan 2x) = \cot x(\tan 3x - \tan 2x)(1 + \tan x \tan 2x) \\ &= \cot^2 x(\tan 3x - \tan 2x)(\tan 2x - \tan x) \\ &= \cot^2 x \left(\frac{\sin x}{\cos 2x \cos 3x}\right)\left(\frac{\sin x}{\cos x \cos 2x}\right) \ . \end{aligned}$$

This has no solution.

Solution 6. For a solution, neither $2x$ nor $3x$ can be a multiple of $\pi/2$, so we exclude these cases. Since

$$\tan 4x = \frac{2\tan 2x}{1 - \tan^2 2x} \ ,$$

we find that

$$\cot 4x = \frac{1 - \tan^2 2x}{2\tan 2x} = -\tan 3x \ ,$$

whence $1 + \tan 3x \tan 4x = 0$. Now

$$\tan 4x - \tan 3x = (1 + \tan 3x \tan 4x)\tan x = 0 \ ,$$

so that $4x \equiv 3x \pmod{\pi}$. But we have excluded this. Hence there is no solution to the equation.