Solutions

325. Solve for positive real values of x, y, t:

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2 y^2 \ .$$

Are there infinitely many solutions for which the values of x, y, t are all positive integers? What is the smallest value of t for a positive integer solution?

Solution. Considering the equation as a quadratic in t, we find that the solution is given by

$$t = \frac{(x^2 + y^2)[x + \sqrt{x^2 + y^2}]}{y^2} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2} - x} \ .$$

where x and y are arbitrary real numbers. The choice of sign before the radical is governed by the condition that $t > 0$. Integer solutions are those obtained by selecting $(x, y) = (k(m^2 - n^2), 2kmn)$ for integers m, n, k where m and n are coprime and k is a multiple of $2n^2$. Then

$$t = \frac{k(m^2 + n^2)^2}{2n^2} \ .$$

The smallest solution is found by taking $n = 1$, $m = k = 2$ to yield $(x, y, t) = (6, 8, 25)$.

Comment. L. Fei gives the solution set

$$(x, y, t) = (2n^2 + 2n, 2n + 1, (2n^2 + 2n + 1)^2) \ .$$

This set of solutions satisfies in particular $y^2 = 2x + 1$. If, in the above solution, one takes $(x, y) = (2mn, m^2 - n^2)$, then $t = (m^2 + n^2)^2/(m - n)^2$; in particular, this gives the solution $(x, y, t) = (4, 3, 25)$.

326. In the triangle ABC with semiperimeter $s = \frac{1}{2}(a + b + c)$, points U, V, W lie on the respective sides BC, CA, AB. Prove that

$$s < |AU| + |BV| + |CW| < 3s \ .$$

Give an example for which the sum in the middle is equal to $2s$.

Solution. The triangle inequality yields that $|AB| + |BU| > |AU|$ and $|AC| + |CU| > |AU|$. Adding these and dividing by 2 gives $s > |AU|$. Applying the same inequality to BV and CW yields that

$$3s > |AU| + |BV| + |CW| \ .$$

Again, by the triangle inequality, $|AU| + |BU| > |AB|$ and $|AU| + |CU| > |AC|$. Adding these inequalities gives $2|AU| > |AB| + |AC| - |BC|$. Adding this to analogous inequalities for BV and CW and dividing by 2 yields that $|AU| + |BV| + |CW| > s$.

Comment. For the last part, most students gave a degenerate example in which the points U, V, W coincided with certain vertices of the triangle. A few gave more interesting examples. However, it was necessary to make clear that the stated lengths assigned to AU, BV and CW were indeed possible, *i.e.* they were at least as great as the altitudes. The nicest example came from D. Dziabenko: $|AB| = 6$, $|BC| = 8$, $|CA| = 10$, $|AU| = 8$, $|BV| = 7$ $|CW| = 9$.

327. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

Solution 1. [Y. Zhao] When $\angle CBO = \angle DBA = 90°$, the result is obvious. Wolog, suppose that $\angle CBO = \angle DBA < 90°$. Suppose that the circumcircle of triangle OBD meets the given circle at G. Since $OBDG$ is concyclic and triangle OGD is isosceles,

$$\angle OBC = \angle ABD = 180° - \angle OBD = \angle OGD = \angle ODG = \angle OBG ,$$

so that $G = C$ and $OBDC$ is concyclic.

Let H lie on OA produced so that $OA = AH$. Since $OB \cdot OH = OA^2$, the inversion in the given circle with centre O interchanges B and H, fixes C and D, and carries the circle $OBDC$ (which passes through the centre O of inversion) to a straight line passing through H, D, C. Thus C, D, H are collinear.

This means that CD always passes through the point H on OA produced for which $OA = AH$. Since E is the midpoint of CD, a chord of the circle with centre O, $\angle OEH = \angle OED = 90°$. Hence E lies on the circle with centre A and radius OA.

Consider the reflection in the point B (the dilation with centre B and factor -1). This takes the circle with centre O and radius OA to the circle with centre A and the same radius, and also interchanges E and F. Since E is on the latter circle, F is on the given circle.

Ad (b), E lies on the arc of the circle with centre A and radius OA that joins O to the point R of intersection of this circle and the given circle. Since $RB \perp OA$, and $OA = OR = RA$, $\angle RAO = 60°$. It can be seen that $\angle EAO$ ranges from $0°$ (when CD is a diameter) to $60°$ (when $C = D = R$).

Solution 2. [A. Wice] We first establish a Lemma.

Lemma. Let UZ be an angle bisector of triangle UVW with Z on VW. Then

$$UZ^2 = UV \cdot UW - VZ \cdot WZ .$$

Proof. By the Cosine Law,

$$UV^2 = UZ^2 + VZ^2 - 2UZ \cdot VZ \cos \angle UZW$$

and

$$UW^2 = UZ^2 + WZ^2 + 2UZ \cdot WZ \cos \angle UZW .$$

Eliminating the cosine term yields that

$$UV^2 \cdot WZ + UW^2 \cdot VZ = (UZ^2 + VZ \cdot WZ)(WZ + VZ) .$$

Now,

$$UV : VZ = UW : WZ = (UV + UW) : (VZ + WZ) ,$$

so that

$$UV \cdot WZ = UW \cdot VZ$$

and

$$(UV + UW) \cdot WZ = UW \cdot (WZ + VZ) .$$

Thus

$$\begin{aligned} UV^2 \cdot WZ + UW^2 \cdot VZ &= (UV + UW) \cdot WZ \cdot UV \\ &= (WZ + VZ) \cdot UW \cdot UV . \end{aligned}$$

It follows that $UW \cdot UV = UZ^2 + VZ \cdot WZ$. ♠

Let R be a point on the circle with $BR \perp OA$, S be the intersection of CD in OA produced, and D' be the reflection of D in OA. (Wolog, $\angle CBO < 90°$.) Since SB is an angle bisector of triangle SCD', from the Lemma, we have that

$$BS^2 = SC \cdot SD' - CB \cdot D'B = SC \cdot SD - BR^2 = SC \cdot SD - (SR^2 - BS^2)$$

whence $SC \cdot SD = SR^2$. Using power of a point, we deduce that SR is tangent to the given circle and $OR \perp SR$.

Now

$$(OA + AS)^2 - OR^2 = RS^2 = BR^2 + (AB + AS)^2 = 3AB^2 + AB^2 + 2AB \cdot AS + AS^2$$

from which $4AB \cdot AS = 4AB^2 + 2AB \cdot AS$, whence $AS = 2AB = OA$. Since $OE \perp CD$, E lies on the circle with diameter OS.

Consider the reflection in the point B (dilation in B with factor -1). It interchanges E and F, interchanges O and A, and switches the circles $ADRC$ and $OERS$. Since E lies on the latter circle, F must lie on the former circle, and the desired result (a) follows.

Ad (b), the locus of E is that part of the circle with centre A that lies within the circle with centre O. Angle EAO is maximum when E coincides with R, and minimum when D coincides with A. Since triangle ORA is equilateral, the maximum angle is $60°$ and the minimum angle is $0°$.

Solution 3. [M. Elqars] Let the radius of the circle be r. Let $\angle CBO = \angle DBA = \alpha$, $\angle DOB = \beta$ and $\angle COF = \gamma$. By the Law of Sines, we have that

$$\sin\alpha : \sin(\gamma - \alpha) = OC : OB = OD : OB = \sin(180° - \alpha) : \sin(\alpha - \beta) = \sin\alpha : \sin(\alpha - \beta) \ ,$$

whence $\alpha - \beta = \gamma - \alpha$. Thus $2\alpha = \beta + \gamma$. Therefore,

$$\angle DOC = 180° - (\beta + \gamma) = 180° - 2\alpha = \angle CBD \ .$$

Thus,

$$\angle DOE = \frac{1}{2}\angle DOC = 90° - \alpha \ ,$$

whence $\angle EDO = \alpha$ and $|OE| = r\sin\alpha$.

Observe that $\sin(\alpha - \beta) : \sin\alpha = OB : OD = 1 : 2$, so that $\sin(\alpha - \beta) = \frac{1}{2}\sin\alpha$.

$$\begin{aligned}
|AE|^2 &= |OE|^2 + |OA|^2 - 2|OE||OA|\cos\angle EOA \\
&= r^2\sin^2\alpha + r^2 - 2r^2\sin\alpha\cos(90° - \alpha + \beta) \\
&= r^2[1 + \sin^2\alpha - 2\sin\alpha\sin(\alpha - \beta)] \\
&= r^2[1 + \sin^2\alpha - \sin^2\alpha] = r^2 \ .
\end{aligned}$$

The segments EF and OA bisect each other, so they are diagonals of a parallelogram $OFAE$. Hence $|OF| = |AE| = r$, as desired. As before, we see that $\angle EAO$ ranges from $0°$ to $60°$.

Solution 4. Assign coordinates: $O \sim (0,0)$, $B \sim (\frac{1}{2}, 0)$, $C \sim (1,0)$, and let the slope of the lines BC and BD be respectively $-m$ and m. Then $C \sim (\frac{1}{2} + s, -ms)$ and $D \sim (\frac{1}{2} + t, mt)$ for some s and t. Using the fact that the coordinates of C and D satisfy $x^2 + y^2 = 1$, we find that

$$C \sim \left(\frac{m^2 - \sqrt{3m^2+4}}{2(m^2+1)}, \frac{-m^3 + m\sqrt{3m^2+4}}{2(m^2+1)}\right)$$

$$D \sim \left(\frac{m^2 + \sqrt{3m^2+4}}{2(m^2+1)}, \frac{m^3 + m\sqrt{3m^2+4}}{2(m^2+1)} \right)$$

$$E \sim \left(\frac{m^2}{2(m^2+1)}, \frac{m\sqrt{3m^2+4}}{2(m^2+1)} \right)$$

$$F \sim \left(\frac{m^2+2}{2(m^2+1)}, \frac{-m\sqrt{3m^2+4}}{2(m^2+1)} \right) .$$

It can be checked that the coordinates of F satisfy the equation $x^2 + y^2 = 1$ and the result follows.

Solution 5. [P. Shi] Assign the coordinates $O \sim (0, -1)$, $B \sim (0, 0)$, $A \sim (0, 1)$. Taking the coordinates of C and D to be of the form $(x, y) = (r \cos\theta, r \sin\theta)$ and $(x, y) = (s \cos\theta, -s \sin\theta)$ and using the fact that both lie of the circle of equation $x^2 + (y + 1)^2 = 4$, we find that

$$C \sim ((\sqrt{\sin^2\theta + 3} - \sin\theta)\cos\theta, (\sqrt{\sin^2\theta + 3} - \sin\theta)\sin\theta)$$

$$D \sim ((\sqrt{sin^2\theta + 3} + \sin\theta)\cos\theta, -(\sqrt{\sin^2\theta + 3} + \sin\theta)\sin\theta)$$

$$E \sim ((\sqrt{\sin^2\theta + 3})\cos\theta, -\sin^2\theta)$$

$$F \sim (-(\sqrt{\sin^2\theta + 3})\cos\theta, \sin^2\theta) .$$

It is straightforward to verify that $|OF|^2 = 4$, from which the result follows.

328. Let $\mathcal{C}$ be a circle with diameter AC and centre D. Suppose that B is a point on the circle for which $BD \perp AC$. Let E be the midpoint of DC and let Z be a point on the radius AD for which $EZ = EB$.

Prove that

(a) The length c of BZ is the length of the side of a regular pentagon inscribed in $\mathcal{C}$.

(b) The length b of DZ is the length of the side of a regular decagon (10-gon) inscribed in $\mathcal{C}$.

(c) $c^2 = a^2 + b^2$ where a is the length of a regular hexagon inscribed in $\mathcal{C}$.

(d) $(a + b) : a = a : b$.

Comment. We begin by reviewing the trigonetric functions of certain angles. Since

$$\begin{aligned} \cos 72^\circ &= 2\cos^2 36^\circ - 1 = 2\cos^2 144^\circ - 1 \\ &= 2(2\cos^2 72^\circ - 1)^2 = 8\cos^4 72^\circ - 8\cos^2 72^\circ + 1 , \end{aligned}$$

$t = \cos 72^\circ$ is a root of the equation

$$0 = 8t^4 - 8t^2 - t + 1 = (t - 1)(2t + 1)(4t^2 + 2t - 1) .$$

Since it must be the quadratic factor that vanishes when $t = \cos 72^\circ$, we find that

$$\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5} - 1}{4} .$$

Hence $\cos 144^\circ = -(\sqrt{5} + 1)/4$ and $\cos 36^\circ = (\sqrt{5} + 1)/4$.

Solution. [J. Park] Wolog, suppose that the radius of the circle is 2.

(a) Select W on the arc of the circle joining A and B such that $BW = BZ$. We have that $|BE| = |EZ| = \sqrt{5}$, $b = |ZD| = \sqrt{5} - 1$, $c^2 = |BZ|^2 + |BW|^2 = 10 - 2\sqrt{5}$ and, by the Law of Cosines applied to triangle BDW,

$$\cos \angle BDW = \frac{1}{4}(\sqrt{5} - 1) .$$

Hence BW subtends an angle of $72°$ at the centre of the circle and so BW is a side of an inscribed regular pentagon.

(b) The angle at each vertex of a regular decagon is $144°$. Thus, the triangle formed by the side of an inscribed regular pentagon and two adjacent sides of an inscribed regular decagon has angles $144°, 18°, 18°$. Conversely, if a triangle with these angles has its longest side equal to that of an inscribed regular pentagon, then its two equal sides have lengths equal to those of the sides of a regular decagon inscribed in the same circle.

Now $b = \sqrt{5} - 1$, $c^2 = 10 - 2\sqrt{5}$, so $4b^2 - c^2 = 14 - 2\sqrt{45} > 0$. Thus, $c < 2b$, and we can construct an isosceles triangle with sides b, b, c. By the Law of Cosines, the cosine of the angle opposite c is equal to $(2b^2 - c^2)/(2b^2) = -\frac{1}{4}(\sqrt{5} - 1)$. This angle is equal to $144°$, and so b is the side length of a regular inscribed decagon.

(c) The side length of a regular inscribed hexagon is equal to the radius of the circle. We have that

$$a^2 = |BD|^2 = |BZ|^2 - |ZD|^2 = c^2 - b^2 \ .$$

(d) Since $b^2 + 2b - 4 = (b+1)^2 - 5 = 0$, $a^2 = 4 = (2+b)b = (a+b)b$, whence $(a+b) : a = a : b$.

329. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} \ .$$

Solution 1. Let ABC be a triangle for which $|AB| = x$, $|AC| = z$ and $\angle BAC = 120°$. Let AD be a ray through A that bisects angle BAC and has length y. By the law of cosines applied respectively to triangle ABC, ABD and ACD, we find that

$$|BC| = \sqrt{x^2 + xz + z^2} \ ,$$
$$|BD| = \sqrt{x^2 - xy + y^2} \ ,$$
$$|CD| = \sqrt{y^2 - yz + z^2} \ .$$

Since $|BD| + |CD| \geq |BC|$, the desired result follows.

Solution 2. [B. Braverman; B.H. Deng] Note that $x^2 - xy + y^2$, $y^2 - yz + z^2$ and $z^2 + xz + z^2$ are always positive [why?]. By squaring, we see that the given inequality is equivalent to

$$x^2 + z^2 + 2y^2 - xy - yz + 2\sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)} \geq x^2 + xz + z^2$$

which reduces to

$$2\sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)} \geq xy + yz + zx - 2y^2 \ .$$

If the right side is negative, then the inequality holds trivially. If the right side is positive, the inequality is equivalent (by squaring) to

$$4(x^2 - xy + y^2)(y^2 - yz + z^2) \geq (xy + yz + zx - 2y^2)^2 \ .$$

Expanding and simplifying gives the equivalent inequality

$$x^2y^2 + x^2z^2 + y^2z^2 - 2x^2yz - 2xyz^2 + 2xy^2z \geq 0$$

or $(xy + yz - zx)^2 \geq 0$. Since the last always holds, the result follows.

Comment. The above write-up proceeds by a succession of equivalent inequalities to one that is trivial, a working-backwards from the result. The danger of this approach is that one may come to a step where the reasoning is not necessarily reversible, so that instead of a chain of equivalent statements, you get to a stage where the logical implication is in the wrong direction. The possibility that $xy + yz + zx < 2y^2$ complicates the argument a litle and needs to be dealt with. To be on the safe side, you could frame the solution by starting with the observation that $(xy + yz - zx)^2 \geq 0$ and deducing that

$$(xy + yz + zx - 2y^2)^2 \leq 4(x^2 - xy + y^2)(y^2 - yz + z^2) \ .$$

From this, we get that

$$xy + yz + zx - 2y^2 \leq |xy + yz + zx - 2y^2| \leq \sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)} \ ,$$

from which the required inequality follows by rearranging the terms between the outside members and taking the square root.

Solution 3. [L. Fei] Let $x = ay$ and $z = by$ for some positive reals a and b. Then the given inequality is equivalent to

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} \geq \sqrt{a^2 + ab + b^2}$$

which in turn (by squaring) is equivalent to

$$2 - a - b + 2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1} \geq ab \ . \qquad (*)$$

We have that

$$\begin{aligned} 0 &\leq 3(ab - a - b)^2 = 3a^2b^2 - 6a^2b - 6ab^2 + 3a^2 + 3b^2 + 6ab \\ &= 4(a^2b^2 - a^2b + a^2 - ab^2 + ab - a + b^2 - b + 1) - (a^2b^2 + a^2 + b^2 + 4 + 2a^2b + 2ab^2 - 2ab - 4a - 4b) \\ &= [2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1}]^2 - (ab + a + b - 2)^2 \ . \end{aligned}$$

Hence

$$2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1} \geq |ab + a + b - 2| \geq ab + a + b - 2 \ .$$

Taking the inequality of the outside members and rearranging the terms yields $(*)$.

Comment. The student who produced this solution worked backwards down to the obvious inequality $(ab - a - b)^2 \geq 0$, However, for a proper argument, you need to show how by logical steps you can go in the other direction, *i.e.* from the obvious inequality to the desired one. You will notice that this has been done in the write-up above; the price that you pay is that the evolution from $(ab - a - b)^2 \geq 0$ seems somewhat artificial. There is a place where care is needed. It is conceivable that $ab + a + b - 2$ is negative, so that $2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1} \geq ab + a + b - 2$ is always true, even when $4(a^2 - a + 1)(b^2 - b + 1) \geq (ab + a + b - 1)^2$ fails. The inequality $A^2 \geq B^2$ is equivalent to $A \geq B$ only if you know that A and B are both positive.

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

Solution 1. Let the four languages be E, F, G, I. If anyone speaks only one language, then everyone else must speak that language, and the result holds. Suppose there is an individual who speaks exactly two languages, say E and F. Then everyone else must speak at least one of E and F. If 60% of the participants speaks a particular one of these languages, then the result holds. Otherwise, at least 40% of the participants, constituting set A, must speak E and not F, and 40%, constituting set B, must speak F and not E. Since each person in A must communicate with each person in B, each person in either of these sets must speak G or I. At least half the members of A must speak a particular one of these latter languages, say G. If any of them speaks only G (as well as E), then everyone in B must speak G and so at least 20% + 40% = 60% of the participants speak G. The remaining possibility is that everyone in A speaks both G and I. At least half the members of B speaks a particular one of these languages, say G, and so 40% + 20% = 60% of the participants speak G. Thus, if anyone speaks only two languages, the result holds.

Finally, suppose that every participant speaks at least three languages. Let $p\%$ speak E, F and G (and possibly, but not necessarily I), $q\%$ speak E, F, I but not G, $r\%$ speak E, G, I but not F, $s\%$ speak F, G, I but not E. Then $p + q + r + s = 100$ and so

$$(p + q + r) + (p + q + s) + (p + r + s) + (q + r + s) = 300 .$$

At least one of the four summands on the left is at least 75, Suppose it is $p + r + s$, say. Then at least 75% speaks G. The result holds once again.

Solution 2. [D. Rhee] As in Solution 1, we take the languages to be E, F, G, I, and can dispose of the case where someone speaks exactly one of these. Let $(A \cdots C)$ denote the set of people who speaks the languages $A \cdots C$ and no other language, and suppose that each person speaks at least two languages.

We first observe that in each of the pairs of sets, $\{(EF), (GI)\}$, $\{(EG), (FI)\}$, $\{(EI), (FG)\}$, at least one of the sets in each pair is empty. So either there is one language that is spoken by everyone speaking exactly two languages, or else there are only three languages spoken among those that speak exactly two languages. Thus, wolog, we can take the two language sets among the participants to be either $\{(EF), (EG), (EI)\}$ or $\{(EF), (FG), (EG)\}$.

Case 1. Everyone is in exactly one of the language groups

$$(EF), (EG), (EI), (EFG), (EFI), (EGI), (FGI), (EFGI) .$$

If no more than 40% of the participants are in (FGI), then at least 60% of the participants speak E. Otherwise, more than 40% of the participants are in (FGI). If no more that 40% of the participants are in $(EG) \cup (EI) \cup (EGI)$, then at least 60% of the participants speak F. The remaining case is that more that 40% of the participants are in each of (FGI) and $(EG) \cup (EI) \cup (EGI)$. It follows that, either at least 20% of the participants are in $(EG) \cup (EGI)$, in which case at least 60% speak G, or at least 20% of the participants are in $(EI) \cup (EGI)$, in which case at least 60% speak I.

Case 2. Everyone is in exactly one of the language groups

$$(EF), (EG), (FG), (EFG), (EFI), (EGI), (FGI), (EFGI) .$$

Since the three sets $(EF) \cup (EFI)$, $(EG) \cup (EGI)$ amd $(FG) \cup (FGI)$ are disjoint, one of them must include fewer than 40% of the participants. Suppose, say, it is $(EF) \cup (EFI)$. Then more than 60% must belong to its complement, and each of these must speak G. The result follows.

331. Some checkers are placed on various squares of a $2m \times 2n$ chessboard, where m and n are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.

Solution 1. Rearrange the rows so that the m odd-numbered rows move into the top m positions and the m even-numbered rows move into the bottom m positions, while all the columns remain intact except for order of entries. Now move all the n odd-numbered columns to the left n positions and all the n even-numbered columns to the right n positions, while the rows remain intact except for order of entries. The conditions of the problem continue to hold. Now the chessboard consists of two diagonally opposite $m \times n$ arrays of black squares and two diagonally opposite $m \times n$ arrays of white squares. Let a and b be the number of checkers in the top $m \times n$ arrays of black and white squares respectively, and c and d be the number of checkers in the arrays of white and black squares respectively. Since each row has an odd number of checkers, and m is odd, then $a + b$ is odd. By a similar argument, $b + d$ is odd. Hence

$$a + d = (a + b) + (b + d) - 2b$$

must be even. But $a + d$ is the total number of checkers on the black squares, and the result follows.

Solution 2. [F. Barekat] Suppose that the $(1,1)$ square on the chessboard is black. The set of black squares is contained in the union U of the odd-numbered columns along with the union V of all the even-numbered rows. Note that U contains an odd number of columns and V an odd number of rows. Since each row and each column contains an odd number of checkers, U has an odd number u of checkers and V has an odd number v of checkers. Thus $u+v$ is even. Note that each white square belongs either to both of U and V, or to neither of them. Thus, the $u+v$ is equal to the number of black checkers plus twice the number of checkers on the white squares common to U and V. The result follows.