PROBLEMS FOR OCTOBER 2004

Please send your solution to
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no later than December 5, 2004. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

339. Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 \ .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

341. Let s, r, R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in geometric progression.

342. Prove that there are infinitely many solutions in positive integers of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 \ .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 \ , \quad a_1 = 1 \ , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

344. A function f defined on the positive integers is given by

$$f(1) = 1 \ , \quad f(3) = 3 \ , \quad f(2n) = f(n) \ ,$$

$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n) \ ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

345. Let $\mathcal{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathcal{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.