PROBLEMS FOR MARCH

Please send your solution to
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no later than April 25, 2004. It is important that your complete mailing address and your email address appear on the front page.

297. The point P lies on the side BC of triangle ABC so that $PC = 2BP$, $\angle ABC = 45°$ and $\angle APC = 60°$. Determine $\angle ACB$.

298. Let O be a point in the interior of a quadrilateral of area S, and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \ .$$

Prove that $ABCD$ is a square with centre O.

299. Let $\sigma(r)$ denote the sum of all the divisors of r, including r and 1. Prove that there are infinitely many natural numbers n for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever $1 \leq k \leq n$.

300. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90°$, and let K be its circumcircle. Suppose that the tangent to K at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular for A to BE and Y the midpoint of AX. Suppose that BY meets K again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.

301. Let $d = 1, 2, 3$. Suppose that M_d consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference d. Prove that, if $c \in M_3$, then there exist integers $a \in M_1$ and $b \in M_2$ for which $c = ab$.

302. In the following, $ABCD$ is an arbitrary convex quadrilateral. The notation $[\cdots]$ refers to the area.

(a) Prove that $ABCD$ is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD] \ .$$

(b) Suppose that F is an interior point of the quadrilateral $ABCD$ such that $ABCF$ is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD] \ .$$

303. Solve the equation

$$\tan^2 2x = 2 \tan 2x \tan 3x + 1 \ .$$