Solutions

255. Prove that there is no positive integer that, when written to base 10, is equal to its kth multiple when its initial digit (on the left) is transferred to the right (units end), where $2 \leq k \leq 9$ and $k \neq 3$.

Solution 1. Note that the number of digits remains the same after multiplication. Thus, if $k \geq 5$, the left digit of the number must be 1 and so the multiple must end in 1. This is impossible for $k = 5, 6, 8$. If $k = 7$ or 9, then the number must have the form $10^m + x$ where $x \leq 10^n - 1$. Then $k(10^m + x) = 10x + 1$, so that

$$x = \frac{k \cdot 10^m - 1}{10 - k} \geq \frac{7 \cdot 10^m - 1}{3} > 2 \times 10^m ,$$

an impossibility.

If $k = 4$, the first digit of the number cannot exceed 2, and so must be even to achieve an even product. Thus, for some positive integers m and $x \leq 10^m - 1$, we must have $4(2 \times 10^m + x) = 10x + 2$, whence

$$x = \frac{4 \times 10^m - 1}{3} > 10^m ,$$

again an impossibility. Finally, if $k = 2$, then $d \leq 4$ and $2(d \cdot 10^m + x) = 10x + d$, whence $d(2 \cdot 10^m - 1) = 8x$. Since $2 \cdot 10^m - 1$ is odd, 8 must divide d, which is impossible. The desired result follows.

Solution 2. [A. Critch] Suppose that multiplication is positive for some $k \neq 3$. Let the number be $d \cdot 10^m + u$ for a positive digit d, a positive integer m and a nonnegative integer $u < 10^m - 1$. Then $k(d \cdot 10^m + u) = 10u + d$, whence

$$(10^m - 1)k < k \cdot 10^m - 1 \leq d(k \times 10^m - 1) = (10 - k)u \leq (10 - k)(10^m - 1) ,$$

so that $k < 10 - k$ and k is equal to 2 or 4. Since k is even, d must be even. Since

$$10 - k = d\left(\frac{k \times 10^m - 1}{u}\right) > d\frac{k \times 10^m - k}{10^m - 1} = dk ,$$

$d < (10/k) - 1$. When $k = 2$, d must be 2, and we get $2(2 \times 10^m - 1) = 8u$, or $2 \times 10^m - 1 = 4u$, an impossibility. When $k = 4$, we get $d < 1.5$, which is also impossible. Hence the multiplication is not possible.

Comment. When $k = 3$, the first digit must be 1, 2 or 3. It can be shown that 2 and 3 do not work, so that we must have $3(10^m + x) = 10x + 1$ for $x = (3 \times 10^m - 1)/7$. This actually gives a result when $m \equiv 5 \pmod 6$. Indeed, when $m = 5$, we obtain the example 142857.

256. Find the condition that must be satisfied by y_1, y_2, y_3, y_4 in order that the following set of six simultaneous equations in x_1, x_2, x_3, x_4 is solvable. Where possible, find the solution.

$$x_1 + x_2 = y_1 y_2 \qquad x_1 + x_3 = y_1 y_3 \qquad x_1 + x_4 = y_1 y_4$$
$$x_2 + x_3 = y_2 y_3 \qquad x_2 + x_4 = y_2 y_4 \qquad x_3 + x_4 = y_3 y_4 .$$

Solution. We have than $y_1(y_2 - y_3) = x_2 - x_3 = y_4(y_2 - y_3)$, whence $(y_1 - y_4)(y_2 - y_3) = 0$. Similarly, $(y_1 - y_2)(y_3 - y_4) = 0 = (y_1 - y_3)(y_2 - y_4)$. From this, we deduce that three of the four y_i must be equal. Suppose, wolog, that $y_1 = y_2 = y_3 = u$ and $y_4 = v$. Then the system can be solved to obtain $x_1 = x_2 = x_3 = u^2/2$ and $x_4 = uv - (u^2/2) = \frac{1}{2}u(2v - u)$. (This includes the case $u = v$.)

257. Let n be a positive integer exceeding 1. Discuss the solution of the system of equations:

$$ax_1 + x_2 + \cdots + x_n = 1$$

$$x_1 + ax_2 + \cdots + x_n = a$$

$$\cdots$$

$$x_1 + x_2 + \cdots + ax_i + \cdots + x_n = a^{i-1}$$

$$\cdots$$

$$x_1 + x_2 + \cdots + x_i + \cdots + ax_n = a^{n-1} \ .$$

Solution 1. First, suppose that $a = 1$. Then all of the equations in the system become $x_1 + x_2 + \cdots + x_n = 1$, which has infinitely many solutions; any $n-1$ of the x_i's can be chosen arbitrarily and the remaining one solved for.

Henceforth, assume that $a \neq 1$. Adding all of the equations leads to

$$(n-1+a)(x_1 + x_2 + \cdots + x_n) = 1 + a + a^2 + \cdots + a_{n-1} = \frac{1-a^n}{1-a} \ .$$

If $a = 1-n$, then the system is viable only if $a^n = 1$. This occurs, only if $a = -1$ and n is a positive integer *i.e.*, when $(n,a) = (2,-1)$. In this case, both equations in the system reduce to $x_2 - x_1 = 1$, and we have infinitely many solution. Otherwise, when $a = 1-n$, there is no solution to the system.

When $a \neq 1-n$, then

$$x_1 + x_2 + \cdots + x_n = \frac{1-a^n}{(1-a)(n-1+a)} \ .$$

Taking the difference between this and the ith equation in the system leads to

$$(a-1)x_i = a^{i-1} - \left(\frac{1-a^n}{(1-a)(n-1+a)} \right)$$

for each i and the system is solved.

Solution 2. As above, we dispose first of the case $a = 1$. Suppose that $a \neq 1$.Taking the difference of adjacent equations leads to $(a-1)(x_{i+1} - x_i) = a^i - a^{i-1}$, so that $x_{i+1} = x_i + a^{i-1}$ for $1 \leq i \leq n-1$. Hence $x_i = x_1 + (1 + a + \cdots + a^{i-2})$ for $2 \leq i \leq n$. From the first equation, we find that

$$\begin{aligned}
(n-1+a)x_1 + 1 + (1+a) + (1+a+a^2) + \cdots + \cdots (1 + a + \cdots + a^{n-2}) = 1 \\
\Longrightarrow (n-1+a)x_1 + \frac{(1-a^2) + \cdots + (1-a^{n-1})}{1-a} = 0 \\
\Longrightarrow (n-1+a)x_1 + \frac{n-2-a^2(1+a+\cdots+a^{n-3})}{1-a} = 0 \\
\Longrightarrow (n-1+a)x_1 + \frac{(n-2)(1-a) - a^2(1-a^{n-2})}{(1-a)^2} = 0 \ .
\end{aligned}$$

Suppose that $n = 1-a$. Then

$$0 = (n-2)(1-a) - a^2(1-a^{n-2}) = -(1+a)(1-a) - a^2(1-a^{n-2}) = a^{n-2} - 1 \ ,$$

so that a must be -1 and $n = 2$, The system reduces to a single equation with an infinitude of solutions. If $n \neq 1-a$, then we can solve for x_1 and then obtain the remaining values of the x_i.

Comment. Beware of the "easy" questions. Many solvers had only a superficial analysis which did not consider the possibility that a denominator might vanish, and almost nobody picked up the $(n,a) = (2,-1)$ case. When you write up your solution, it is good to dispose of the singular cases first before you get into the general situation.

258. The infinite sequence $\{a_n; n = 0, 1, 2, \cdots\}$ satisfies the recursion

$$a_{n+1} = a_n^2 + (a_n - 1)^2$$

for $n \geq 0$. Find all rational numbers a_0 such that there are four distinct indices p, q, r, s for which $a_p - a_q = a_r - a_s$.

Solution. The recursion can be rewritten as

$$a_{n+1} = 2a_n^2 - 2a_n + 1 \Leftrightarrow 2a_{n+1} - 1 = (2a_n - 1)^2 \ .$$

Let $b_n = 2a_n - 1$, so that $a_n = \frac{1}{2}(b_n + 1)$. Then $a_p - a_q = a_r - a_s$ is equivalent to $b_p - b_q = b_r - b_s$. Since $b_{n+1} = b_n^2$ for each nonnegative integer n, we have that $b^n = b_0^{2^n}$. If $b_p - b_q = b_r - b_s$, then b_0 must be the rational solution of a polynomial equation of the form,

$$x^{2^p} - x^{2^q} - x^{2^r} + x^{2^s} = 0$$

where the left side consists of four distinct monomials. One possibility is $b_0 = 0$. Suppose now that $b_0 \neq 0$. Dividing by the monomial with the smallest exponent, we obtain a polynomial equation for b_0 whose leading coefficient and constant coefficients are each 1. So the numerator of b_0 written in lowest terms, dividing the constant term, must be ± 1 and the denominator, dividing the leading coefficient, must also be ± 1. Hence, the only possibilities for b_0 are -1, 0 and 1. These correspond to the possibilities $0, \frac{1}{2}, 1$ for a_0, and each of these choices leads to a sequence for which $a_n = a_1$ for $n \geq 1$ and for which there are two pairs of terms with the same difference (0).

259. Let ABC be a given triangle and let $A'BC$, $AB'C$, ABC' be equilateral triangles erected outwards on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the respective intersections of Ω with the lines AA', BB', CC'.

Prove that AA', BB', CC' are concurrent and that

$$AA'' + BB'' + CC'' = AA' = BB' = CC' \ .$$

Solution. A rotation of $60°$ about the vertex A takes triangle ACC' to the triangle $AB'B$, and so $BB' = CC'$. Similarly, it can be shown that each of these is equal to AA'. Suppose that BB' and CC' intersect in F. From the rotation, $\angle BFC' = 60° = \angle BAC'$, so that $AFBC'$ is concyclic.

hence $\angle C'FB = \angle C'AB = 60°$. Also $\angle AFC' = \angle ABC' = 60°$, $\angle AFB' = 60°$ and so $\angle BFC = \angle C'FB' = 120°$. Since $\angle BFC + \angle BA'C = 180°$, the quadrilateral $BFCA'$ is concyclic and $\angle BFA' = \angle BCA' = 60°$. Hence $\angle AFA' = \angle AFC' + \angle C'FB + \angle BFA' = 180°$, so that A, A' and F are collinear, and AA', BB' and CC' intersect at F.

From Ptolemy's Theorem, $AB \cdot C'F = AF \cdot BC' + FB \cdot AC'$, whence $C'F = AF + BF$. Similarly, $A'F = BF + CF$ and $C'F = AF + BF$. Indeed, $AA' = BB' = CC' = AF + A'F = AF + BF + CF$.

[J. Zhao] Let O be the circumcentre of triangle $A'B'C'$ and let the respective midpoints of $A'A''$, $B'B''$, $C'C''$ be X, Y, Z. Since $OX \perp A'A''$, $OX \perp FX$. Similarly, $OY \perp FY$ and $OZ \perp FZ$, so that X, Y, Z lie on the circle with diameter OF. Suppose, wolog, that F lies on the arc ZX. Then $\angle XZY = \angle XFY = \angle A'FB'' = 60°$ and $\angle ZXY = \angle ZFY = 60°$, so that XYZ is an equilateral triangle and Ptolemy's theorem yields that $FY = FX + FZ$.

Hence

$$\begin{aligned} AA'' + BB'' + CC'' &= (A'A'' + B'B'' + C'C'') - (AA' + BB' + CC') \\ &= 2(A'X + B'Y + C'Z) - (AA' + BB' + CC') \\ &= 2(A'X \pm FX + B'Y \mp FY + C'Z \pm FZ) - (AA' + BB' + CC') \\ &= 2(A'F + B'F + C'F) - (AA' + BB' + CC') \\ &= 4(AF + BF + CF) - 3(AF + BF + CF) \\ &= AF + BF + CF = AA' = BB' = CC' \ . \end{aligned}$$

260. $TABC$ is a tetrahedron with volume 1, G is the centroid of triangle ABC and O is the midpoint of TG. Reflect $TABC$ in O to get $T'A'B'C'$. Find the volume of the intersection of $TABC$ and $T'A'B'C'$.

Solution. Denote by X' the reflection of a point X in O. In particular, $T' = G$. Let D be the midpoint of BC. Since $TT' = TG$ and AA' intersect at O, the points A, G, D, T, A' are collinear. Let A_1 be the intersection of DT and GA'. Since the reflection in O takes any line to a parallel line, $A'G \| AT$, so that (from triangle DTA), $DA_1 : DT = DG : DA = 1 : 3$ and A_1 is the centroid of triangle TBC. Also

$$GA_1 : GA' = GA_1 : AT = DA_1 : DT = 1 : 3$$

so that $GA_1 = (1/3)GA'$.

Applying the same reasoning all around, we see that each side of one tetrahedron intersects a face of the other in its centroid one third of the way along its length. Thus GA' intersects TBC in A_1, GB' intersects TAC in B_1, GC' intersects TAB in C_1, TA intersects $GB'C'$ in A_2, TB intersects $GA'C'$ in B_2 and TC intersects $GA'B'$ in C_2. Note that the $A'_i = A_j$, $B'_i = B_j$, $C'_i = C_j$ for $i \neq j$.

The intersection of the two tetrahedra is a parallelepiped with vertices $T, A_2, B_2, C_2, A_1, B_1, C_1, G$ and faces $TA_2C_1B_2$, $TB_2A_1C_2$, $TC_2B_1A_2$, $GA_1C_2B_1$, $GB_1A_2C_1$, $GC_1B_2A_1$ (to see that, say, $TB_2A_1C_2$ is a parallelogram, note that a dilation with centre T and factor $3/2$ takes it to a parallelogram with diagonal TD). The volume of this parallelpiped is three times that of the skew pyramid $TB_2A_1C_2A_2$ with base $TB_2A_1C_2$ and altitude dropped from A_2, which in turn is twice that of tetrahedron $TA_2B_2C_2$. But the volume of tetrahedron $TA_2B_2C_2$ is $1/27 = (1/3)^3$ that of $TABC$ since it can be obtained from $TABC$ by a dilation with centre T and factor $1/3$. Hence the volume of the parallelpiped common to both tetraheda $TABC$ and $GA'B'C'$ is $6 \times (1/27) = 2/9$ is the volume of either of these tetrahedra.

261. Let $x, y, z > 0$. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(x+y)(y+z)}} + \frac{z}{z + \sqrt{(x+z)(y+z)}} \leq 1 \ .$$

Solution. Observe that

$$(x+y)(x+z) - (\sqrt{xy} + \sqrt{xz})^2 = x^2 + yz - 2x\sqrt{yz} = (x - \sqrt{yz})^2 \geq 0$$

(with equality iff $x^2 = yz$). Hence

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \frac{x}{x + \sqrt{xy} + \sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}} \ ,$$

with a similar inequality for the other two terms on the left side. Adding these inequalities together leads to the desired result.