Solutions

228. Prove that, if $1 < a < b < c$, then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 \ .$$

Solution. Since $1 < a < b < c$, $\log_a b > 1$, so that

$$\log_a(\log_a b) = \log_a b \cdot \log_b(\log_a b) > \log_b(\log_a b) \ .$$

Also

$$0 < \log_c a = \log_c b \cdot \log_b a < \log_c b < 1 \ ,$$

so that $\log_b(\log_c a) < 0$ and

$$\log_c(\log_c a) = \log_c b \cdot \log_b(\log_c a) > \log_b(\log_c a) \ .$$

Hence,

$$\begin{aligned}\log_a(\log_a b) &+ \log_b(\log_b c) + \log_c(\log_c a) \\ &> \log_b(\log_a b) + \log_b(\log_b c) + \log_b(\log_c a) \\ &= \log_b(\log_a b \cdot \log_b c \cdot \log_c a) = \log_b 1 = 0 \ .\end{aligned}$$

Comment. As an exercise, you should justify the following fundamental facts about change of basis, beginning with the definition, $\log_p q = r$ iff $p^r = q$ where $0 < p, q$ and $p \neq 1$: (1) $\log_u v \cdot \log_v w = \log_u w$; (2) $\log_u v = 1/(\log_v u)$.

229. Suppose that n is a positive integer and that $0 < i < j < n$. Prove that the greatest common divisor of $\binom{n}{i}$ and $\binom{n}{j}$ exceeds 1.

First solution. Since $\binom{n}{k} = \binom{n}{n-k}$ for $1 \leq k \leq n-1$, it suffices to prove the result when $0 < i < j \leq n/2$, so that $i + j \leq n$. Observe that

$$\binom{n}{i} = \left(\frac{n}{n-i}\right)\left(\frac{n-1}{n-1-i}\right)\cdots\left(\frac{n-j+1}{n-j+1-i}\right)\binom{n-j}{i}$$

so that $\binom{n}{i} > \binom{n-j}{i}$, and that

$$\binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!j!(n-i-j)!} = \binom{n}{j}\binom{n-j}{i} \ .$$

Suppose, if possible, that $\binom{n}{i}$ and $\binom{n}{j}$ are coprime. Then, since $\binom{n}{i}$ divides the product of $\binom{n}{j}$ and $\binom{n-j}{i}$, $\binom{n}{i}$ must divide $\binom{n-j}{i}$. But this is impossible, since $\binom{n-j}{i} < \binom{n}{i}$.

Second solution. Observe that, for $1 \leq i < j \leq n-1$,

$$\binom{n}{i} = \frac{n(n-1)\cdots(j+1)}{(n-i)\cdots(j-i+1)}\binom{j}{i} > \binom{j}{i}$$

and

$$\binom{n}{i}\binom{n-i}{j-i} = \frac{n!}{i!(j-i)!(n-j)!} = \binom{n}{j}\binom{j}{i} \ .$$

If $\binom{n}{i}$ and $\binom{n}{j}$ were coprime, then $\binom{n}{i}$ would divide the smaller $\binom{j}{i}$, an impossibility.

230. Let f be a strictly increasing function on the closed interval $[0,1]$ for which $f(0)=0$ and $f(1)=1$. Let g be its inverse. Prove that
$$\sum_{k=1}^{9}\left(f\left(\frac{k}{10}\right)+g\left(\frac{k}{10}\right)\right)\leq 9.9\ .$$

Solution. Observe that $x=g(y)$ and $y=f(x)$ determine the same curve. Sketch a diagram that includes the graph of $y=f(x)$ and the rectangles with vertices $(k/10, f(k/10))$, $(k/10,0)$, $((k+1)/10,0)$, $((k+1)/10, f(k/10)$ and areas $(1/10)f(k/10)$, for $1\leq k\leq 0$. The area under the graph of $y=f(x)$ and the $x-$axis for $1/10\leq x\leq 1$ is at least $(1/10)\sum_{k=1}^{9}f(k/10)$.

Similarly, the area between the graph of $x=g(y)$ and the $y-$axis for $1/10\leq y\leq 1$ is at least $(1/10)\sum_{k=1}^{9}g(k/10)$. Since both these regions do not overlap the square with side $1/10$ and opposite vertices at $(0,0)$ and $(1/10,1/10)$, we must have
$$\frac{1}{100}+\frac{1}{10}\sum_{k=1}^{9}\left(f\left(\frac{k}{10}\right)+g\left(\frac{k}{10}\right)\right)\leq 1$$

from which the result follows.

231. For $n\geq 10$, let $g(n)$ be defined as follows: n is mapped by g to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example, $g(54)=918$ since $54^2=2916$ and $2+916=918$. Is it possible to start with 527 and, through repeated applications of g, arrive at 605?

Solution. Suppose $n\geq 1000$. Then $g(n)\geq\lfloor n^2/1000\rfloor\geq 1000$. Since $g(527)=1006$, the result of each subsequent repeated application of g also exceeds 1000 and so can never be 605.

232. (a) Prove that, for positive integers n and positive values of x,
$$(1+x^{n+1})^n\leq(1+x^n)^{n+1}\leq 2(1+x^{n+1})^n\ .$$

(b) Let $h(x)$ be the function defined by
$$h(x)=\begin{cases}1, & \text{if } 0\leq x\leq 1;\\ x, & \text{if } x>1.\end{cases}$$

Determine a value N for which
$$|h(x)-(1+x^n)^{\frac{1}{n}}|<10^{-6}$$
whenever $0\leq x\leq 10$ and $n\geq N$.

Solution. Recall the power-mean inequality
$$\left(\frac{a^n+b^n}{2}\right)^{1/n}\leq\left(\frac{a^{n+1}+b^{n+1}}{2}\right)^{1/(n+1)}$$

for n a positive integer and $a,b>0$. Applying this to $(a,b)=(1,x)$ yields
$$(1+x^n)^{n+1}\leq 2(1+x^{n+1})^n\ .$$

If $0\leq x\leq 1$, then $x^{n+1}\leq x^n$ and
$$(1+x^{n+1})^n\leq(1+x^n)^n\leq(1+x^n)^{n+1}\ .$$

Let $1 \le x$. Then

$$\left(1 + \frac{1}{x^{n+1}}\right)^n \le \left(1 + \frac{1}{x^n}\right)^n .$$

Multiplying by $x^{n(n+1)}$ yields $(x^{n+1}+1)^n \le (x^n+1)^{n+1}$, as desired.

(b) Let $0 \le x \le 1$. Then, for each positive integer n, $|1 - (1+x^n)^{1/n}| = (1+x^n)^{1/n} - 1 \le 2^{1/n} - 1$. Now let $1 \le x \le 10$. Then, for each positive integer n,

$$|x - (1+x^n)^{1/n}| = (1+x^n)^{1/n} - x \le (2x^n)^{1/n} - x = x[2^{1/n} - 1] \le 10(2^{1/n} - 1) .$$

It follows that, for $0 \le x \le 10$ and each positive integer n,

$$|h(x) - (1+x^n)^{1/n}| \le 10(2^{1/n} - 1) .$$

Suppose that N is an integer that exceeds $1/\log_2(1+10^{-7})$. (N could be 3×10^7 for example.) Then

$$\begin{aligned} n \ge N &\Longrightarrow \frac{1}{n} \le \frac{1}{N} < \log_2(1+10^{-7}) \\ &\Longrightarrow 2^{1/n} < (1+10^{-7}) \Longrightarrow 10(2^{1/n} - 1) < 10^{-6} \\ &\Longrightarrow |h(x) - (1+x^n)^{1/n}| < 10^{-6} \end{aligned}$$

for $0 \le x \le 10$.

Comments. The (b) part of this question was badly handled, and solvers did not make the logic of the situation clear. This is a situation, where one works backwards to determine what a suitable value of N might be. Unfortunately, this working backwards involves starting with the desired result, and so the implications are in reverse. For problems of this type, the solution **must** be re-edited to put it into the proper logical form: start with what is given; proceed by justified logical steps to what is desired. The appropriate final form of the solution thus should be: "Let N be equal to $\cdots$. Then (following a sequence of manipulations), $|h(x) - \cdots|$&c." Note that in the above solution, we do a little initial spadework to get an upper bound independent of x for the difference. Having gotten the upper bound, we then define a suitable value of N. The final part of the solution then shows that N does the job, using material that is already known to be true. Those of you who will be studying mathematics at university will undoubtedly in their initial analysis course encounter $\epsilon - \delta$ arguments, which are notoriously difficult for many students to grasp. The present solution is such an argument for a particular value of ϵ, so an attempt to really understand the logical structure at this point will pay dividends for you later on.

There are other ways of establishing (a). For example, when $0 \le x \le 1$,

$$\left(\frac{1+x^{n+1}}{1+x^n}\right)^n \le 1 \le 1 + x^n$$

while, if $1 \le x$,

$$\begin{aligned} (x^n+1)^{n+1} - (x^{n+1}+1)^n &= \sum_{r=0}^{n} \left[\binom{n+1}{r} x^{n(n+1-r)} - \binom{n}{r} x^{(n+1)(n-r)}\right] + 1 \\ &= \sum_{r=0}^{n} \left[\binom{n+1}{r} x^r - \binom{n}{r}\right] x^{(n+1)(n-r)} + 1 \ge 1 . \end{aligned}$$

One could use the Arithmetic-Geometric Means Inequality to obtain

$$\begin{aligned} (1+x^{n+1})^2 \le (1+x^n)(1+x^{n+2}) &\Rightarrow (1+x^{n+1})^{2(n+1)} \le (1+x^n)^{n+1}(1+x^{n+2})^{n+1} \\ &\Rightarrow \frac{(1+x^{n+1})^{n+2}}{(1+x^{n+2})^{n+1}} \le \frac{(1+x^n)^{n+1}}{(1+x^{n+1})^n} \end{aligned}$$

for each positive integer n. When $n = 1$, we have

$$\frac{(1+x)^2}{1+x^2} = 1 + \frac{2x}{1+x^2} \leq 2$$

from which one of the inequalities follows.

233. Let $p(x)$ be a polynomial of degree 4 with rational coefficients for which the equation $p(x) = 0$ has *exactly one* real solution. Prove that this solution is rational.

Solution. Suppose that $p(x) = x^4 + tx^3 + ux^2 + vx + w$, where t, u, v, w are all rational. (There is no loss of generality in supposing that the leading coefficient is 1.) Since $p(x) = 0$ has exactly one real solution r and since nonreal solutions come in pairs, there are two possibilities: (a) $p(x) = (x-r)^4$, in which case $r = -t/4$ is rational, or (b) r is a double root and $p(x) = (x-r)^2(x^2+bx+c) = x^4 + (b-2r)x^3 + (c-2br+r^2)x^2 + (br^2-2cr)x + cr^2$. Then $t = b - 2r$, $u = c - 2br + r^2$, $v = br^2 - 2cr$ and $w = cr^2$.

We find that $4r^3 + 3tr^2 + 2ur + v = 0$ (by manipulating the values for t, &c, to eliminate b and c), so that r is a root of the cubic polynomial

$$q(x) = 4x^3 + 3tx^2 + 2ux + v$$

with rational coefficients. Hence, r is a root of the quadratic

$$f(x) \equiv 16p(x) - (4x+t)q(x) = (8u - 3t^2)x^2 + (12v - 2ut)x + (16w - vt)$$

with rational coefficients. (Use long division to divide $q(x)$ into $p(x)$.) It is not possible for all coefficients of $f(x)$ to vanish, for this would imply that $u = (3/8)t^2$, $v = (3/48)t^3$, $w = (1/4^4)t^4$ and $p(x) = (x + (t/4))^4$, nor can f be a nonzero constant. If $8u - 3t^2 = 0$, then $r = -(16w - vt)/(12v - 2ut)$ is rational. If $8u - 3t^2 \neq 0$, then we can divide $q(x)$ by $p(x)$ to get a relation $q(x) = f(x)g(x) + h(x)$, where $h(x)$ is a linear polynomial with rational coefficients and the root r. In this case, also, r is rational.

Comment. If you have knowledge of calculus, then you can note that $p(x) = x^4 + tx^3 + ux^2 + vx + w = (x-r)^2(x^2+bx+c)$ implies that

$$\begin{aligned} q(x) = p'(x) &= 4x^3 + 3tx^2 + 2ux + v \\ &= 2(x-r)(x^2+bx+c) + (x-r)^2(2x+b) \\ &= (x-r)[2(x^2+bx+c) + (x-r)(2x+b)] \end{aligned}$$

so that both $p(x)$ and $q(x)$ have root r, We can proceed as above to get a linear polynomial with root r.