Solutions

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.

Solution 1. The statement is false. To see how to obtain the solution, start with a triangle XYZ with $\angle XYZ < \angle XZY < 90^\circ$. Then it is possible to find a point W on YZ for which $XW = XZ$ (this is the diagram for the *ambiguous case* ASS-congruence situation). There are two ways of gluing a copy of triangle XYW to XYZ (the copy of XW glued along XZ) to give a quadrilateral with an opposite pair of angles equal to $\angle Y$ and an opposite pair of sides equal to $|XY|$. One of these satisfies the condition and is not a parallelogram.

C. Shen followed this strategy with $|XY| = 8$, $\angle XYZ = 60^\circ$, $|YW| = 3$ and $|YZ| = 5$ to obtain a quadrilateral $ABCD$ with $|AB| = 5$, $|BC| = 8$, $|CD| = 3$, $|DA| = 8$, $|BD| = 7$ and $\angle DAB = \angle DCB = 60^\circ$.

Solution 2. The statement is false. Suppose that we have fixed D, A, B and that AB is one of the equal sides and $\angle DAB$ is one of the equal angles. Then C is the intersection of two circles. One of the circles contains the locus of points at which DB subtends an angle equal to $\angle DAB$ and the other circle is that with centre D and radius equal to $|AB|$. The two circles are either tangent or have two points of intersection. One of these points will give the expected parallelogram, so the question arises whether the other point will give a suitable quadrilateral. We show that it can.

Using coordinate geometry, we may take $A \sim (0,0)$, $B \sim (3,0)$, $D \sim (2,2)$ so that $\angle DAB = 45^\circ$. The point E that completes the parallelogram is $(5,2)$, and this will be one of the intersections of the two circles. The circle that subtends an angle of 45° from DB has as its centre the circumcentre of ΔBDE, namely $(7/2, 3/2)$; this circle has equation $x^2 - 7x + y^2 - 3y + 12 = 0$. The circle with centre D and radius $3 = |AB|$ has equation $x^2 - 4x + y^2 - 4y - 1 = 0$. These circles intersect at the points $E \sim (5,2)$ and $C \sim (22/5, 1/5)$. The quadrilateral $ABCD$ satisfies the given conditions but is not a parallelogram.

Comment. Investigate what happens when A, B and D are assigned the coordinates $(0,0)$, $(2,0)$ and (i) $(1,1)$ or (ii) $(2,2)$, respectively.

Comment. Consider the following two "proofs" that the quadrilateral must be a parallelogram.

"Proof" 1. Let $AB = CD$ and $\angle A = \angle C$. Suppose that X and Y, respectively, are the feet of the perpendiculars dropped from B to AD and from D to BC. Then triangles AXB and CYD, having equal acute angles and equal hypotenuses must be congruent. Hence $AX = CY$, and also $BX = DY$, from which it can be deduced that triangles BXD and DYB are congruent. Therefore $XD = YB$ and so $AD = BC$ and the quadrilateral is a parallelogram.

"Proof" 2. Suppose that $AB = CD$ and that $\angle B = \angle D$. Applying the Law of Sines, we find that

$$\frac{DC}{\sin \angle DAC} = \frac{AC}{\sin \angle ADC} = \frac{AC}{\sin \angle ABC} = \frac{AB}{\sin \angle ACB} = \frac{CD}{\sin \angle ACB} \ .$$

Therefore, $\angle DAC = \angle ACB$ so that $\angle DCA = \angle BAC$ and $AB \| DC$.

221. A *cycloid* is the locus of a point P fixed on a circle that rolls without slipping upon a line u. It consists of a sequence of arches, each arch extending from that position on the locus at which the point P rests on the line u, through a curve that rises to a position whose distance from u is equal to the diameter of the generating circle and then falls to a subsequent position at which P rests on the line u. Let v be the straight line parallel to u that is tangent to the cycloid at the point furthest from the line u.

(a) Consider a position of the generating circle, and let P be on this circle and on the cycloid. Let PQ be the chord on this circle that is parallel to u (and to v). Show that the locus of Q is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line v.

(b) The region between the two cycloids consists of a number of "beads". Argue that the area of one of these beads is equal to the area of the generating circle.

(c) Use the considerations of (a) and (b) to find the area between u and one arch of the cycloid using a method that does not make use of calculus.

Solution. (a) Suppose the circle generating the cycloid rotates from left to right. We consider half the arc of the cycloid joining a point T to a point W on v. Let P be an intermediate point on the cycloid and Q be the point on the generating circle as described in the problem. Suppose that the perpendicular dropped from W to u meets u at Y and the perpendicular dropped from T to v meets v at X. Thus $TXWY$ is a rectangle with $|TX| = |WY| = 2r$ and $|TY| = |XW| = \pi r$, where r is the radius of the generating circle.

Let the generating circle touch u and v at U and V, respectively. Then $|\text{arc } (PU)| = |TU|$, so that

$$|\text{arc } VQ| = |\text{arc } VP| = \pi r - |\text{arc } PU| = \pi r - |TU| = |UY| = |VW| \ .$$

This means that Q is on the circle of radius r rolling to the left generating a second cycloid passing through W, Q, T. This second cycloid is the image of the first under a $180°$ rotation that interchanges the points T and W.

(b, c) Let α be the area of the region within the rectangle $TXWY$ bounded by the two cycloids (one of the "beads"), β be the area above the cycloid TPW and γ the area below the cycloid TQW within the rectangle. Because the region $TXVWP$ is congruent to the region $WYUTQ$, $\beta = \gamma$. Hence

$$\alpha + 2\beta = \alpha + \beta + \gamma = (2r)(\pi r) = 2\pi r^2 \ .$$

At each vertical height between the lines u and v, the length of the chord PQ of the "bead" is equal to the length of the chord at the same height of the generating circle, so that the "bead" can be regarded as being made of infinitesimal slats of the circle that have been translated. Thus, the "bead" has the same area as the generating circle, namely πr^2 (this is due to a principle enunciated by a seventeenth century mathematician, Cavalieri). Thus $\alpha = \pi r^2$ and $2\beta = 2\pi r^2 - \alpha = \pi r^2$. The area under the cycloid and above TY is equal to $\alpha + \beta$ and the area under a complete arch of the cycloid is $2\alpha + 2\beta = 2\pi r^2 + \pi r^2 = 3\pi r^2$, three times the area of the generating circle.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

Solution 1. Let $a_n = \tan^{-1} n$ for $n \geq 0$. Thus, $0 < a_n < \pi/2$ and $\tan a_n = n$. Then

$$\tan(a_{n+1} - a_{n-1}) = \frac{(n+1) - (n-1)}{1 + (n^2 - 1)} = \frac{2}{n^2}$$

for $n \geq 1$. Then

$$\sum_{n=1}^{m} \tan^{-1} \frac{2}{n^2} = \tan^{-1}(m+1) + \tan^{-1} m - \tan^{-1} 1 - \tan^{-1} 0 \ .$$

Letting $m \to \infty$ yields the answer $\pi/2 + \pi/2 - \pi/4 - 0 = 3\pi/4$.

Solution 2. Let $b_n = \tan^{-1}(1/n)$ for $n \geq 0$. Then

$$\tan(b_{n-1} - b_{n+1}) = \frac{2}{n^2}$$

for $n \geq 2$, whence

$$\begin{aligned}\sum_{n=1}^{m} \tan^{-1} \frac{2}{n^2} &= \tan^{-1} 2 + \sum_{n=2}^{m} (b_{n-1} - b_{n+1}) = \tan^{-1} 2 + \tan^{-1} 1 + \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1} \\ &= (\tan^{-1} 2 + \cot^{-1} 2) + \tan^{-1} 1 - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1} \\ &= \frac{\pi}{2} + \frac{\pi}{4} - \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+1}\end{aligned}$$

for $m \geq 3$, from which the result follows by letting m tend to infinity.

Solution 3. [S. Huang] Let $s_n = \sum_{k=1}^n \tan^{-1}(2/n^2)$ and $t_n = \tan s_n$. Then $\{t_n\} = \{2, \infty, -9/2, -14/5. -20/9, \cdots\}$ where the numerators of the fractions are $\{-2, -5, -9, -14, -20, \cdots\}$ and the denominators are $\{-1, 0, 2, 5, 8, \cdots\}$. We conjecture that

$$t_n = \frac{-n(n+3)}{(n-2)(n+1)}$$

for $n \geq 1$. This is true for $1 \leq n \leq 5$. Suppose that it holds to $n = k-1 \geq 5$, so that $t_{k-1} = -(k-1)(k+2)/(k-3)k$. Then

$$\begin{aligned} t_k &= \frac{t_{k-1} + (2/k^2)}{1 - 2t_{k-1}k^{-2}} \\ &= \frac{-k^2(k-1)(k+2) + 2(k-3)k}{k^3(k-3) + 2(k-1)(k+2)} \\ &= \frac{-k(k+3)(k^2-2k+2)}{(k-2)(k+1)(k^2-2k+2)} = \frac{-k(k+3)}{(k-2)(k+1)} \ . \end{aligned}$$

The desired expression for t_n holds by induction and so $\lim_{n\to\infty} t_n = -1$. For $n \geq 3$, $t_n < 0$ and $\tan^{-1}(2/n^2) < \pi/2$, so we must have $\pi/2 < s_n < \pi$ and $s_n = \pi - \tan^{-1} t_n$. Therefore

$$\lim_{n\to\infty} s_n = \tan^{-1}(\pi + \lim_{n\to\infty} t_n) = \pi - (\pi/4) = (3\pi)/4 \ .$$

223. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} \ .$$

Solution 1. Let $a = \tan\alpha$, $b = \tan\beta$, $c = \tan\beta$, where $\alpha, \beta, \gamma \in (0, \pi/2)$. Then

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha} = \frac{a+b+c-abc}{1-ab-bc-ca} = 0 \ ,$$

whence $\alpha + \beta + \gamma = \pi$. Then, the left side of the inequality is equal to

$$\begin{aligned} \cos\alpha + \cos\beta + \cos\gamma &= \cos\alpha + \cos\beta - \cos(\alpha+\beta) \\ &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1 \\ &\leq 2\cos\left(\frac{\alpha+\beta}{2}\right) - 2\cos^2\left(\frac{\alpha+\beta}{2}\right) + 1 \\ &= 2\sin\left(\frac{\gamma}{2}\right) - 2\sin^2\left(\frac{\gamma}{2}\right) + 1 \\ &= \frac{3}{2} - \frac{1}{2}(2\sin(\gamma/2) - 1)^2 \leq \frac{3}{2} \ , \end{aligned}$$

with equality if and only if $\alpha = \beta = \gamma = \pi/3$.

Solution 2. Define α, β and γ and note that $\alpha + \beta + \gamma = \pi$ as in Solution 1. Since $\cos x$ is a concave function on $[0, \pi/2]$, we have that

$$\frac{\cos\alpha + \cos\beta + \cos\gamma}{3} \leq \cos\left(\frac{\alpha+\beta+\gamma}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2} \ ,$$

from which the result follows.

Solution 3. [G. N. Tai] Define α, β, γ as in Solution 1 and let $s = \cos\alpha + \cos\beta + \cos\gamma$. Then

$$s = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2} = 2\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\sin^2\frac{\gamma}{2} \ .$$

Thus, for each α. β, the quadratic equation

$$2t^2 - 2\cos\frac{\alpha-\beta}{2}\cdot t + (s-1) = 0$$

has at least one real solution, namely $t = \sin(\gamma/2)$. Hence, its discriminant is positive, so that

$$\cos^2\frac{\alpha-\beta}{2} - 2(s-1) \geq 0 \Longrightarrow 2s \leq 2 + \cos^2\frac{\alpha-\beta}{2} \leq 3 \Longrightarrow s \leq 3/2 \ .$$

Equality occurs if and only if $\alpha = \beta = \gamma = \pi/3$.

224. For $x > 0$, $y > 0$, let $g(x,y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x,y)$ and where this maximum is assumed.

Solution 1. When $(x,y) = (\sqrt{2}, 1/\sqrt{2})$, all three functions $x, y + (1/x), 1/y$ assume the value $\sqrt{2}$ and so $g(\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}$.

If $0 < x \leq \sqrt{2}$, then $g(x,y) \leq x \leq \sqrt{2}$. Suppose that $x \geq \sqrt{2}$. If $y \geq 1/\sqrt{2}$, then $g(x,y) \leq 1/y \leq \sqrt{2}$. If $0 < y \leq 1/\sqrt{2}$, then

$$g(x,y) \leq y + (1/x) \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \ .$$

Thus, when $x > 0$, $y > 0$, then $g(x,y) \leq \sqrt{2}$. If either $x \neq \sqrt{2}$ or $y \neq 1/\sqrt{2}$, then the foregoing inequalities lead to $g(x,y) < \sqrt{2}$. Hence $g(x,y)$ assumes its maximum value of $\sqrt{2}$ if and only if $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 2. [M. Abdeh-Kolachi] Let u be the minimum of x, $y + (1/x)$ and $1/y$. Then $u \leq x$, $u \leq 1/y$ and $u \leq y + (1/x)$. By the first two inequalities, we also have than $y + (1/x) \leq (1/u) + (1/u) = 2/u$, so that $u \leq 2/u$ and $u \leq \sqrt{2}$. Hence $g(x,y) \leq \sqrt{2}$ for all $x, y > 0$. Since $g(\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}$, g has a maximum value of $\sqrt{2}$ assumed when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

We need to verify that this maximum is assumed nowhere else. Suppose that $g(x,y) = \sqrt{2}$. Then $\sqrt{2} \leq x$, $\sqrt{2} \leq 1/y$ and

$$\sqrt{2} \leq y + (1/x) \leq (1/\sqrt{2}) + (1/\sqrt{2}) = \sqrt{2} \ .$$

We must have equality all across the last inequality and this forces both x and $1/\sqrt{y}$ to equal $\sqrt{2}$.

Solution 3. [R. Appel] If $x \leq 1$ and $y \leq 1$, then $g(x,y) \leq x \leq 1$. If $y \geq 1$, then $g(x,y) \leq 1/y \leq 1$. It remains to examine the case $x > 1$ and $y < 1$, so that $y + (1/x) < 2$. Suppose that $\min\,(x, 1/y) = a$ and $\max\,(x, 1/y) = b$. Then $\min\,(1/x, y) = 1/a$ and $\max\,(1/x, y) = 1/b$, so that

$$y + \frac{1}{x} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \ .$$

Hence $g(x,y) = \min\,(a, (a+b)/(ab))$. Either $a^2 \leq 2$ or $a^2 \geq 2$. But in the latter case,

$$\frac{a+b}{ab} \leq \frac{2b}{\sqrt{2}b} = \sqrt{2} \ .$$

In either case, $g(x,y) \leq \sqrt{2}$. This maximum value is attained when $(x,y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 4. [D. Varodayan] By the continuity of the functions, each of the regions $\{(x,y) : 0 < x < y + (1/x), xy < 1\}$, $\{(x,y) : 0 < x, y + (1/x) < x, y + (1/x) < (1/y)\}$, and $\{(x,y) : 0 < (1/y) < x, (1/y) < y + (1/x)\}$ is an open subset of the plane; using partial derivatives, we see that none of the three functions being minimized have any critical values there. It follows that any extreme values of $g(x,y)$ must occur on one of the curves defined by the equations

$$x = y + (1/x) \tag{1}$$

$$x = 1/y \tag{2}$$

$$y + (1/x) = (1/y) \tag{3}$$

On the curve (1), $x > 1$ and

$$\begin{aligned} g(x,y) &= \min\left(x, \frac{x}{x^2-1}\right) \\ &= \begin{cases} x, & \text{if } x \leq \sqrt{2}; \\ \frac{x}{x^2-1}, & \text{if } x \geq \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (2),

$$\begin{aligned} g(x,y) &= \min\ (x, 2/x) \\ &= \begin{cases} x, & \text{if } x \leq \sqrt{2}; \\ 2/x, & \text{if } x \geq \sqrt{2}. \end{cases} \end{aligned}$$

On the curve (3), $0 < y < 1$ and

$$\begin{aligned} g(x,y) &= \min\left(\frac{y}{1-y^2}, \frac{1}{y}\right) \\ &= \begin{cases} \frac{y}{1-y^2}, & \text{if } 0 < y < \frac{1}{\sqrt{2}}; \\ 1/y, & \text{if } \frac{1}{\sqrt{2}} \leq y \leq 1. \end{cases} \end{aligned}$$

On each of these curves, $g(x,y)$ reaches its maximum value of $\sqrt{2}$ when $(x,y) = (\sqrt{2}.1/\sqrt{2})$.

Solution 5. [J. Sparling] Let $z = 1/y$. For fixed z, let

$$v_z(x) = \min\ \{x, z, (1/x) + (1/z)\}$$

and

$$w(z) = \max\ \{v_z(x) : x > 0\}\ .$$

Suppose that $z \leq 1$. Then $(1/x) + (1/z) \geq z$, so $v_z(x) = \min\ \{x, z\}$ and

$$v_z(x) = \begin{cases} x, & \text{for } x \leq z; \\ z, & \text{for } x \geq z; \end{cases}$$

so that $w(z) = z$ when $z \leq 1$. Suppose that $1 < z \leq \sqrt{2}$, so that $z \leq z/(z^2-1)$. Then

$$v_z(x) = \begin{cases} x, & \text{for } x \leq z; \\ z, & \text{for } z \leq x < z/(z^2-1); \\ (1/x) + (1/z), & \text{for } z/(z^2-1) \leq x; \end{cases}$$

so that $w(z) = z$ when $1 < z \leq \sqrt{2}$. Finally, suppose that $\sqrt{2} > z$. Note that $x \leq (1/x) + (1/z) \Leftrightarrow zx^2 - x - z \leq 0$. Then the minimum of x and $(1/x) + (1/z)$ is x when $zx^2 - x - z \leq 0$, or $x \leq (1 + \sqrt{1+4z^2})/2z$. Since

$$\begin{aligned} \sqrt{2} - \left[\frac{1+\sqrt{1+4z^2}}{2z}\right] &= \frac{(2\sqrt{2}z - 1) - \sqrt{1+4z^2}}{2z} \\ &= \frac{4z^2 - 4\sqrt{2}z}{2z[(2\sqrt{2}z-1) + \sqrt{1+4z^2}]} \\ &= \frac{2(z-\sqrt{2})}{(2\sqrt{2}z-1) + \sqrt{1+4z^2}} \geq 0\ , \end{aligned}$$

this minimum is always less than z, so that

$$v_z(x) = \begin{cases} x, & \text{for } x \le \frac{1+\sqrt{1+4z^2}}{2z} \\ \frac{1}{x} + \frac{1}{z}, & \text{for } x \ge \frac{1+\sqrt{1+4z^2}}{2z}, \end{cases}$$

so that $w(z) = (1 + \sqrt{1+4z^2})/(2z) \le \sqrt{2}$ when $\sqrt{2} \le z$. Hence, the minimum value of $w(z) = \sqrt{2}$ and this is the maximum value of $g(x, y)$, assumed when $(x, y) = (\sqrt{2}, 1/\sqrt{2})$.

Solution 6. For $x > 0$, let

$$h_x(y) = \min\left(x, y + \frac{1}{x}, \frac{1}{y}\right) .$$

Suppose that $x \le \sqrt{2}$. Then $x - (1/x) \le (1/x)$ and

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{if } 0 < y \le x - \frac{1}{x}; \\ x, & \text{if } x - \frac{1}{x} \le y \le \frac{1}{x}; \\ \frac{1}{y}, & \text{if } \frac{1}{x} \le y; \end{cases}$$

so that the minimum value of $h_x(y)$ is x, and this occurs when $x - (1/x) \le y \le (1/x)$. Suppose that $x \ge \sqrt{2}$. Then $y + (1/x) \le (1/y) \Leftrightarrow xy^2 + y - x \le 0$ and

$$\begin{aligned} \sqrt{2} - \left[\frac{1+\sqrt{1+4x^2}}{2x}\right] &= \frac{(\sqrt{8}x - 1) - \sqrt{1+4x^2}}{2x} \\ &= \frac{4x^2 - 4\sqrt{2}x}{2x[(\sqrt{8}x - 1) + \sqrt{1+4x^2})} \\ &= \frac{2(x - \sqrt{2})}{(\sqrt{8}x - 1) + \sqrt{1+4x^2}} \ge 0 \ , \end{aligned}$$

so that

$$\frac{1+\sqrt{1+4x^2}}{2x} \le \sqrt{2} \le x \ .$$

$$h_x(y) = \begin{cases} y + \frac{1}{x}, & \text{when } 0 < y \le \frac{-1+\sqrt{1+4x^2}}{2x}; \\ \frac{1}{y}, & \text{when } \frac{-1+\sqrt{1+4x^2}}{2x} \le y; \end{cases}$$

so that the minimum value of $h_x(y)$ is $(1+\sqrt{1+4x^2})/(2x)$, and this occurs when $y = (-1+\sqrt{1+4x^2})/(2x)$.

Thus, we have to maximize the function $u(x)$ where

$$u(x) = \begin{cases} x, & \text{if } 0 < x \le \sqrt{2}; \\ \frac{1+\sqrt{1+4x^2}}{2x}, & \text{if } \sqrt{2} \le x. \end{cases}$$

By what we have shown, this maximum is $\sqrt{2}$ and is attained when $x = \sqrt{2}$. The result follows.

225. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \ge 3$, switch k can turn bulb k on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

Solution. (a) Clearly $x_1 = 1$ and $x_2 = 2$. Let $n \geq 3$. The only way that bulb n can be turned on is for bulb $n-1$ to be off and for bulbs $1, 2, \cdots, n-2$ to be turned on. Once bulb n is turned on, then we need get bulb $n-1$ turned on. The only way to do this is to turn off bulb $n-2$; but for switch $n-2$ to work, we need to have bulb $n-3$ turned off. So before we can think about dealing with bulb $n-1$, we need to get the first $n-2$ bulbs turned off. Then we will be in the same situation as the outset with $n-1$ rather than n bulbs. Thus the process has the following steps: (1) Turn on bulbs $1, \cdots, n-2$; (2) Turn on bulb n; (3) Turn off bulbs $n-2, \cdots, 1$; (3) Turn on bulbs $1, 2, \cdots, n$. So if, for each positive integer k, y_k is the length of the shortest algorithm to turn them off after all are lit, then

$$x_n = x_{n-2} + 1 + y_{n-2} + x_{n-1} \ .$$

We show that $x_n = y_n$ for $n = 1, 2, \cdots$. Suppose that we have an algorithm that turns all the bulbs on. We prove by induction that at each step we can legitimately reverse the whole sequence to get all the bulbs off again. Clearly, the first step is to turn either bulb 1 or bulb 2 on; since the switch is functioning, we can turn the bulb off again. Suppose that we can reverse the first $k-1$ steps and are at the kth step. Then the switch that operates the bulb at that step is functioning and can restore us to the situation at the end of the $(k-1)$th step. By the induction hypothesis, we can go back to having all the bulbs off. Hence, given the bulbs all on, we can reverse the steps of the algorithm to get the bulbs off again. A similar argument allows us to reverse the algorithm that turns the bulbs off. Thus, for each turning-on algorithm there is a turning-off algorithm of equal length, and vice versa. Thus $x_n = y_n$.

We have that $x_n = x_{n-1} + 2x_{n-2} + 1$ for $n \geq 3$. By, induction, we show that, for $m = 1, 2, \cdots$,

$$x_{2m} = 2x_{2m-1} \qquad \text{and} \qquad x_{2m+1} = 2x_{2m} + 1 = 4x_{2m-1} + 1 \ .$$

This is true for $m = 1$. Suppose it is true for $m \geq 1$. Then

$$\begin{aligned} x_{2(m+1)} &= x_{2m+1} + 2x_{2m} + 1 = 2(x_{2m} + 1) + 4x_{2m-1} \\ &= 2(x_{2m} + 2x_{2m-1} + 1) = 2x_{2m+1} \ , \end{aligned}$$

and

$$\begin{aligned} x_{2(m+1)+1} &= x_{2(m+1)} + 2x_{2m+1} + 1 = 2x_{2m+1} + 4x_{2m} + 3 \\ &= 2(x_{2m+1} + 2x_{2m} + 1) + 1 = 2x_{2(m+1)} + 1 \ . \end{aligned}$$

Hence, for $m \geq 1$,

$$3x_{2m+1} + 1 = 4(3x_{2m-1} + 1) = \cdots = 4^m(3x_1 + 1) = 4^{m+1} = 2^{2(m+1)} \ .$$

Thus, the largest prime divisor is 2.

226. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

Solution. Wolog, we may assume that the leading coefficient is 1. Let $f(x) = \prod_{i=1}^k (x - r_i)^{m_i}$, where $n = \sum_{i=1}^k m_i$. Then

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^k \frac{m_i}{x - r_i} \ .$$

Note that the derivative of this function, $-\sum_{i=1}^k m_i (x - r_i)^{-2} < 0$, so that it decreases on each interval upon which it is defined. By considering the graph of $f'(x)/f(x)$, we see that $f'(x)/f(x) \geq 1/\lambda$ on finitely many intervals of the form $(r_i, s_i]$, where $r_i < s_i$ and the r_i and s_j interlace, and $f'(x)/f(x) \leq -1/\lambda$ on finitely many intervals of the form $[t_i, r_i)$, where $t_i < r_i$ and the t_i and r_j interlace. For each i, we have $t_i < r_i < s_i < t_{i+1}$.

The equation $f'(x)/f(x) = 1/\lambda$ can be rewritten as

$$\begin{aligned} 0 =& (x-r_1)(x-r_2)\cdots(x-r_k) - \lambda \sum_{i=1}^{k} m_i (x-r_1)\cdots\widehat{(x-r_i)}\cdots(x-r_k) \\ =& x^k - \left(\sum_{i=1}^{k} r_i - \lambda \sum_{i=1}^{k} m_i \right) x^{k-1} + \cdots \ . \end{aligned}$$

(The "hat" indicates that the term in the product is deleted.) The sum of the roots of this polynomial is

$$s_1 + s_2 + \cdots + s_k = r_1 + \cdots + r_k - \lambda n \ ,$$

so that $\sum_{i=1}^{m}(s_i - r_i) = \lambda n$. This is the sum of the lengths of the intervals $(r_i, s_i]$ on which $f'(x)/f(x) \geq 1/\lambda$. Similarly, we can show that $f'(x)/f(x) \leq -1/\lambda$ on a finite collection of intervals of total length λn. The set on which the inequality of the problem holds is equal to the union of all of these half-open intervals and the set $\{r_1, r_2, \cdots, r_k\}$. The result follows.