PROBLEMS FOR MARCH

Please send your solution to
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no later than April 30, 2003. It is important that your complete mailing address and your email address appear on the front page.

213. Suppose that each side and each diagonal of a regular hexagon $A_1A_2A_3A_4A_5A_6$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, 6$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Prove that

$$\sum_{k=1}^{6}(2r_k - 7)^2 \leq 54 \ .$$

214. Let S be a circle with centre O and radius 1, and let P_i $(1 \leq i \leq n)$ be points chosen on the (circumference of the) circle for which $\sum_{i=1}^{n} \overrightarrow{OP_i} = \mathbf{0}$. Prove that, for each point X in the plane, $\sum |XP_i| \geq n$.

215. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

216. Let x be positive and let $0 < a \leq 1$. Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1}$$

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.

218. Let ABC be a triangle. Suppose that D is a point on BA produced and E a point on the side BC, and that DE intersects the side AC at F. Let $BE + EF = BA + AF$. Prove that $BC + CF = BD + DF$.

219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points P such that the sum of its distances from two fixed points F_1 and F_2 (called *foci*) is constant.

(2) An ellipse is the locus of points P such that, for some real number e (called the *eccentricity*) with $0 < e < 1$, the distance from P to a fixed point F (called a *focus*) is equal to e times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.