PROBLEMS FOR DECEMBER

Please send your solution to
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no later than January 15, 2004. It is important that your complete mailing address and your email address appear on the front page.

276. Let a, b, c be the lengths of the sides of a triangle and let $s = \frac{1}{2}(a+b+c)$ be its semi-perimeter and r be the radius of the inscribed circle. Prove that

$$(s-a)^{-2} + (s-b)^{-2} + (s-c)^{-2} \geq r^{-2}$$

and indicate when equality holds.

277. Let m and n be positive integers for which $m < n$. Suppose that an arbitrary set of n integers is given and the following operation is performed: select any m of them and add 1 to each. For which pairs (m, n) is it always possible to modify the given set by performing the operation finitely often to obtain a set for which all the integers are equal?

278. (a) Show that $4mn - m - n$ can be an integer square for infinitely many pairs (m, n) of integers. Is it possible for either m or n to be positive?

(b) Show that there are infinitely many pairs (m, n) of positive integers for which $4mn - m - n$ is one less than a perfect square.

279. (a) For which values of n is it possible to construct a sequence of abutting segments in the plane to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where two neighbouring segments are perpendicular?

(b) For which values of n is it possible to construct a sequence of abutting segments in space to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where any two of three sucessive segments are perpendicular?

280. Consider all finite sequences of positive integers whose sum is n. Determine $T(n, k)$, the number of times that the positive integer k occurs in all of these sequences taken together.

281. Let a be the result of tossing a black die (a number cube whose sides are numbers from 1 to 6 inclusive), and b the result of tossing a white die. What is the probability that there exist real numbers x, y, z for which $x + y + z = a$ and $xy + yz + zx = b$?

282. Suppose that at the vertices of a pentagon five integers are specified in such a way that the sum of the integers is positive. If not all the integers are non-negative, we can perform the following operation: suppose that x, y, z are three consecutive integers for which $y < 0$; we replace them respectively by the integers $x + y, -y, z + y$. In the event that there is more than one negative integer, there is a choice of how this operation may be performed. Given any choice of integers, and any sequence of operations, must we arrive at a set of nonnegative integers after a finite number of steps?

For example, if we start with the numbers $(2, -3, 3, -6, 7)$ around the pentagon, we can produce $(1, 3, 0, -6, 7)$ or $(2, -3, -3, 6, 1)$.