Solutions

We begin with an old problem that no one managed to solve.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

Determine $f(m)$.

Solution. [N. Sato] The value of $f(m)$ is $2m-1$. The set of $2m-2$ numbers consisting of $m-1$ zeros and $m-1$ ones does not satisfy the property; from this we can see that n cannot be less than $2m-1$.

We first establish that, if $f(u) = 2u-1$ and $f(v) = 2v-1$, then $f(uv) = 2uv-1$. Suppose that $2uv-1$ numbers are given. Select any $2u-1$ at random. By hypothesis, there exists a $u-$subset whose sum is divisible by u; remove these u elements. Continue removing $u-$subsets in this manner until there are fewer than u numbers remaining. Since $2uv-1 = (2v-1)u + (u-1)$, we will have $2v-1$ sets of u numbers summing to a multiple of u. For $1 \leq i \leq 2v-1$, let ua_i be the sum of the ith of these $2v-1$ sets. We can choose exactly v of the a_i whose sum is divisible by v. The v $u-$sets corresponding to these form the desired uv elements whose sum is divisible by uv. Thus, if we can show that $f(p) = 2p-1$ for each prime p, we can use the fact that each number is a product of primes to show that $f(m) = 2m-1$ for each positive integer m.

Let $x_1, x_2, \cdots, x_{2p-1}$ be $2p-1$ integers. Wolog, we can assume that the x_i have been reduced to their least non-negative residue modulo p and that they are in increasing order. For $1 \leq i \leq p-1$, let $y_i = x_{p+i} - x_i$; we have that $y_i \geq 0$. If $y_i = 0$ for some i, then $x_{i+1} = \cdots = x_{p+i}$, in which case $x_{i+1} + \cdots + x_{p+i}$ is a multiple of p and we have achieved our goal. Henceforth, assume that $y_i > 0$ for all i

Let $s = x_1 + x_2 + \cdots + x_p$. Replacing x_i by x_{p+i} in this sum is equivalent to adding y_i. We wish to show that there is a set of the y_i whose sum is congruent to $-s$ modulo p; this would indicate which of the first p x_i to replace to get a sum which is a multiple of p.

Suppose that $A_0 = \{0\}$, and, for $k \geq 1$, that A_k is the set of distinct numbers i with $0 \leq i \leq p-1$ which either lie in A_{k-1} or are congruent to $a + y_k$ for some a in A_{k-1}. Note that the elements of each A_k is equal to 0 or congruent (modulo p) to a sum of distinct y_i. We claim that the number of elements in A_k must increase by at least one for every k until A_k is equal to $\{0, 1, \cdots, p-1\}$.

Suppose that going from A_{j-1} to A_j yields no new elements. Since $0 \in A_{j-1}$, $y_j \in A_j$, which means that $y_j \in A_{j-1}$. Then $2y_j = y_j + y_j \in A_j = A_{j-1}$, $3y_j = 2y_j + y_j \in A_j = A_{j-1}$, and so on. Thus, all multiples of y_j (modulo p) are in A_{j-1}. As p is prime, we find that A_{j-1} must contain $\{0, 1, \cdots, p-1\}$. We deduce that some sum of the y_i is congruent to $-s$ modulo p and obtain the desired result.

145. Let ABC be a right triangle with $\angle A = 90°$. Let P be a point on the hypotenuse BC, and let Q and R be the respective feet of the perpendiculars from P to AC and AB. For what position of P is the length of QR minimum?

Solution. $PQAR$, being a quadrilateral with right angles at A, Q and R, is a rectangle. Therefore, its diagonals QR and AP are equal. The length of QR is minimized when the length of AP is minimized, and this occurs when P is the foot of the perpendicular from A to BC.

Comment. P must be chosen so that $PB : PC = AB^2 : AC^2$.

146. Suppose that ABC is an equilateral triangle. Let P and Q be the respective midpoint of AB and AC, and let U and V be points on the side BC with $4BU = 4VC = BC$ and $2UV = BC$. Suppose that

PV is joined and that W is the foot of the perpendicular from U to PV and that Z is the foot of the perpendicular from Q to PV.

Explain how that four polygons $APZQ$, $BUWP$, $CQZV$ and UVW can be rearranged to form a rectangle. Is this rectangle a square?

Solution. Consider a $180°$ rotation about Q so that C falls on A, Z falls on Z_1 and V falls on V_1. The quadrilateral $QZVC$ goes to QZ_1V_1A, ZQZ_1 is a line and $\angle QAV_1 = 60°$. Similarly, a $180°$ rotation about P takes quadrilateral $PBUW$ to PAU_1W_1 with WPW_1 a line and $\angle U_1AP = 60°$. Since $\angle U_1AP = \angle PAQ = \angle QAV_1 = 60°$, U_1AV_1 is a line and

$$U_1V_1 = U_1A + AV_1 = UB + CV = \frac{1}{2}BC = UV \ .$$

Translate U and V to fall on U_1 and V_1 respectively; let W fall on W_2. Since

$$\angle W_1U_1W_2 = \angle W_1U_1A + \angle W_2U_1A = \angle WUB + \angle WUV = 180° \ ,$$

$$\angle W_2V_1Z_1 = \angle W_2V_1A + \angle AV_1Z_1 = \angle WVU + \angle CVZ = 180° \ ,$$

and

$$\angle W_2 = \angle W_1 = \angle Z_1 = \angle WZQ = 90° \ ,$$

it follows that $Z_1W_2W_1Z$ is a rectangle composed of isometric images of $APZQ$, $BUWP$, $CQZV$ and UVW.

Since PU and QV are both parallel to the median from A to BC, we have that $PQVU$ is a rectangle for which $PU < PB = PQ$. Thus, $PQVU$ is not a square and so its diagonals PV and QU do not intersect at right angles. It follows that W and Z do not lie on QU and so must be distinct.

Since PZQ and VWU are right triangles with $\angle QPZ = \angle UVW$ and $PQ = VU$, they must be congruent, so that $PZ = VW$, $PW = ZV$ and $UW = QZ$. Since

$$\begin{aligned} W_1W_2 &= W_1U_1 + U_1W_2 = WU + UW = WU + QZ \\ &< UQ = PV = PZ + ZV = PZ + PW = PZ + PW_1 = W_1Z \ , \end{aligned}$$

the adjacent sides of $Z_1W_2W_1Z$ are unequal, and so the rectangle is not square.

Comment. The inequality of the adjacent sides of the rectangle can be obtained also by making measurements. Take 4 as the length of a side of triangle ABC. Then

$$|PU| = \sqrt{3} \ , \quad |PQ| = 2 \ , \quad |QU| = |PV| = \sqrt{7} \ .$$

Since the triangles PUW and PVU are similar, $UW : PU = VU : PV$, whence $|UW| = 2\sqrt{21}/7$. Thus, $|W_1W_2| = 4\sqrt{21}/7 \neq \sqrt{7} = |W_1Z|$.

One can also use the fact that the areas of the triangle and rectangle are equal. The area of the triangle is $4\sqrt{3}$. It just needs to be verified that one of the sides of the rectangle is not equal to the square root of this.

147. Let $a > 0$ and let n be a positive integer. Determine the maximum value of

$$\frac{x_1x_2\cdots x_n}{(1+x_1)(x_1+x_2)\cdots(x_{n-1}+x_n)(x_n+a^{n+1})}$$

subject to the constraint that $x_1, x_2, \cdots, x_n > 0$.

Solution. Let $u_0 = x_1$, $u_i = x_{i+1}/x_i$ for $1 \leq i \leq n-1$ and $u_n = a^{n+1}/x_n$. Observe that $u_0u_1\cdots u_n = a^{n+1}$. The quantity in the problem is the reciprocal of

$$\begin{aligned} &(1+u_0)(1+u_1)(1+u_2)\cdots(1+u_n) \\ &\qquad = 1 + \sum u_i + \sum u_iu_j + \cdots + \sum u_{i_1}u_{i_2}\cdots u_{i_k} + \cdots + u_0u_1\cdots u_n \ . \end{aligned}$$

For $k = 1, 2, \cdots, n$, the sum $\sum u_{i_1} u_{i_2} \cdots u_{i_k}$ adds together all the $\binom{n+1}{k}$ $k-$fold products of the u_i; the product of all the terms in this sum is equal to a^{n+1} raised to the power $\binom{n}{k-1}$, namely, to a raised to the power $k\binom{n+1}{k}$. By the arithmetic-geometric means inequality

$$\sum u_{i_1} u_{i_2} \cdots u_{i_k} \geq \binom{n+1}{k} a^k \ .$$

Hence

$$(1+u_0)(1+u_1)\cdots(1+u_n) \geq 1 + (n+1)a + \cdots + \binom{n+1}{k} a^k + \cdots a^{n+1} = (1+a)^{n+1} \ ,$$

with equality if and only if $u_0 = u_1 = \cdots = u_n = a$. If follows from this that the quantity in the problem has maximum value of $(1+a)^{-(n+1)}$, with equality if and only if $x_i = a_i$ for $1 \leq i \leq n$.

Comment. Some of you tried the following strategy. If any two of the u_i were unequal, they showed that a larger value could be obtained for the given expression by replacing each of these by another value. They then deduced that the maximum occurred when all the u_i were equal. There is a subtle difficulty here. What has really been proved is that, **if** there is a maximum, it can occur only when the u_i are equal. However, it begs the question of the existence of a maximum. To appreciate the point, consider the following argument that 1 is the largest postive integer. We note that, given any integer n exceeding 1, we can find another integer that exceeds n, namely n^2. Thus, no integer exceeding 1 can be the largest positive integer. Therefore, 1 itself must be the largest.

Some of you tried a similar approach with the x_i, and showed that for a maximum, one must have all the x_i equal to 1. However, they neglected to build in the relationship between x_n and a_{n+1}, which of course cannot be equal if all the x_i are 1 and $a \neq 1$. This leaves open the possibility of making the given expression larger by bettering the relationship between the x_i and a and possibly allowing inequalities of the variables.

148. For a given prime number p, find the number of distinct sequences of natural numbers (positive integers) $\{a_0, a_1, \cdots, a_n \cdots\}$ satisfying, for each positive integer n, the equation

$$\frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_n} + \frac{p}{a_{n+1}} = 1 \ .$$

Solution. For $n \geq 3$ we have that

$$\begin{aligned} 1 = & \frac{a_0}{a_1} + \cdots + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_{n-2}} + \frac{p}{a_{n-1}} \\ = & \frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_{n-1}} + \frac{p}{a_n} \end{aligned}$$

whence

$$\frac{p}{a_{n-1}} = \frac{a_0}{a_{n-1}} + \frac{p}{a_n} \ ,$$

so that

$$a_n = \frac{pa_{n-1}}{p - a_0} \ .$$

Thus, for $n \geq 2$, we have that

$$a_n = \frac{p^{n-2} a_2}{(p-a_0)^{n-2}} \ .$$

Since $1 \leq p - a_0 \leq p - 1$, $p - a_0$ and p are coprime. It follows that, either $p - a_0$ must divide a_2 to an arbitrarily high power (impossible!) or $p - a_0 = 1$.

Therefore, $a_0 = p - 1$ and $a_n = p^{n-2}a_2$ for $n \geq 2$. Thus, once a_1 and a_2 are selected, then the rest of the sequence $\{a_n\}$ is determined. The remaining condition that has to be satisfied is

$$1 = \frac{a_0}{a_1} + \frac{p}{a_2} = \frac{p-1}{a_1} + \frac{p}{a_2} \ .$$

This is equivalent to

$$(p-1)a_2 + pa_1 = a_1 a_2 \ ,$$

or

$$[a_1 - (p-1)][a_2 - p] = p(p-1) \ .$$

The factors $a_1 - (p-1)$ and $a_2 - p$ must be both negative or both positive. The former case is excluded by the fact that $(p-1) - a_1$ and $p - a_2$ are respectively less than $p - 1$ and p. Hence, each choice of the pair (a_1, a_2) corresponds to a choice of a pair of positive divisors of $p(p-1)$. There are $d(p(p-1)) = 2d(p-1)$ such choices, where $d(n)$ is the number of positive divisors of the positive integer n.

Comment. When $p = 5$, for example, the possibilities for (a_1, a_2) are $(5, 25)$, $(6, 15)$, $(8, 10)$, $(9, 9)$, $(14, 7)$, $(24, 6)$. In general, particular choices of sequences that work are

$$\{p-1, p, p^2, p^3, \cdots\}$$

$$\{p-1, 2p-1, 2p-1, p(2p-1), \cdots\}$$

$$\{p-1, p^2-1, p+1, p(p+1), \cdots\} \ .$$

A variant on the argument showing that the a_n from some point on constituted a geometric progression started with the relation $p(a_n - a_{n-1}) = a_0 a_n$ for $n \geq 3$, whence

$$\frac{a_{n-1}}{a_n} = 1 - \frac{a_0}{p} \ .$$

Thus, for $n \geq 3$, $a_{n+1}a_{n-1} = a_n^2$, which forces $\{a_2, a_3, \cdots, \}$ to be a geometric progession. The common ratio must be a positive integer r for which $r = p/(p - a_0)$. This forces $p - a_0$ to be equal to 1.

Quite a few solvers lost points because of poor book-keeping; they did not identify the correct place at which the geometric progression began. It is often a good idea to write out the first few equations of a general relation explicitly in order to avoid this type of confusion. You must learn to pay attention to details and check work carefully; otherwise, you may find yourself settling for a score on a competition less than you really deserve on the basis of ability.

149. Consider a cube concentric with a parallelepiped (rectangular box) with sides $a < b < c$ and faces parallel to that of the cube. Find the side length of the cube for which the difference between the volume of the union and the volume of the intersection of the cube and parallelepiped is minimum.

Solution. Let x be the length of the side of the cube and let $f(x)$ be the difference between the value of the union and the volume of the intersection of the two solids. Then

$$f(x) = \begin{cases} abc - x^3 & (0 \leq x < a) \\ abc + (x-a)x^2 - ax^2 = abc + x^3 - 2ax^2 & (a \leq x < b) \\ x^3 + ab(c-x) - abx = abc + x^3 - 2abx & (b \leq x < c) \\ x^3 - abc & (c \leq x) \end{cases}$$

The function decreases for $0 \leq x \leq a$ and increases for $x \geq c$. For $b \leq x \leq c$,

$$\begin{aligned} f(x) - f(b) &= x^3 - 2abx - b^3 + 2ab^2 \\ &= (x-b)[x^2 + bx + b^2 - 2ab] \\ &= (x-b)[(x^2 - ab) + b(x-a) + b^2] \geq 0 \ , \end{aligned}$$

so that $f(x) \geq f(b)$. Hence, the minimum value of $f(x)$ must be assumed when $a \leq x \leq b$.

For $a \leq x \leq b$, $f'(x) = 3x^2 - 4ax = x[3x - 4a]$, so that $f(x)$ increases for $x \geq 4a/3$ and decreases for $x \leq 4a/3$. When $b \leq 4a/3$, then $f(x)$ is decreasing on the closed interval $[a, b]$ and assumes its minimum for $x = b$. If $b > 4a/3 > a$, then $f(x)$ increases on $[4a/3, b]$ and so achieves its minimum when $x = 4a/3$. Hence, the function $f(x)$ is minimized when $x = \min(b, 4a/3)$.

150. The area of the bases of a truncated pyramid are equal to S_1 and S_2 and the total area of the lateral surface is S. Prove that, if there is a plane parallel to each of the bases that partitions the truncated pyramid into two truncated pyramids within each of which a sphere can be inscribed, then

$$S = (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 .$$

Solution 1. Let M_1 be the larger base of the truncated pyramid with area S_1, and M_2 the smaller base with area S_2. Let P_1 be the entire pyramid with base M_1 of which the truncated pyramid is a part. Let M_0 be the base parallel to M_1 and M_2 described in the problem, and let its area be S_0. Let P_0 be the pyramid with base M_0 and P_2 the pyramid with base M_2.

The inscribed sphere bounded by M_0 and M_1 is determined by the condition that it touches M_1 and the lateral faces of the pyramid; thus, it is the inscribed sphere of the pyramid P_1 with base M_1; let its radius be R_1. The inscribed sphere bounded by M_2 and M_0 is the inscribed sphere of the pyramid P_0 with base M_0; let its radius be R_0. Finally, let the inscribed sphere of the pyramid with base M_2 have radius R_2.

Suppose Q_2 is the lateral area of pyramid P_2 and Q_1 the lateral area of pyramid P_1. Thus, $S = Q_1 - Q_2$.

There is a dilation with factor R_0/R_1 that takes pyramid P_1 to P_0; since it takes the inscribed sphere of P_1 to that of P_0, it takes the base M_1 to M_0 and the base M_0 to M_2. Hence, this dilation takes P_0 to P_2. The dilation composed with itself takes P_1 to P_2. Therefore

$$\frac{R_0}{R_1} = \frac{R_2}{R_0} \qquad \text{and} \qquad \frac{Q_2}{Q_1} = \frac{S_2}{S_1} = \frac{R_2^2}{R_1^2} .$$

Consider the volume of P_2. Since P_2 is the union of pyramids of height R_2 and with bases the lateral faces of P_2 and M_2, its volume is $(1/3)R_2(Q_2 + S_2)$. However, we can find the volume of P_2 another way. P_2 can be realized as the union of pyramids whose bases are its lateral faces and whose apexes are the centre of the inscribed sphere with radius R_0 with the removal of the pyramid of base M_2 and apex at the centre of the same sphere. Thus, the volume is also equal to $(1/3)R_0(Q_2 - S_2)$.

Hence

$$\frac{Q_2 - S_2}{Q_2 + S_2} = \frac{R_2}{R_0} = \frac{R_2}{\sqrt{R_1 R_2}} = \frac{\sqrt{R_2}}{\sqrt{R_1}} = \frac{\sqrt[4]{S_2}}{\sqrt[4]{S_1}}$$

$$\Longrightarrow Q_2(\sqrt[4]{S_1} - \sqrt[4]{S_2}) = S_2(\sqrt[4]{S_1} + \sqrt[4]{S_2}) ,$$

so that

$$\begin{aligned} S = Q_1 - Q_2 &= \frac{Q_2}{S_2}(S_1 - S_2) \\ &= \left[\frac{\sqrt[4]{S_1} + \sqrt[4]{S_2}}{\sqrt[4]{S_1} - \sqrt[4]{S_2}}\right][\sqrt{S_1} - \sqrt{S_2}][\sqrt{S_1} + \sqrt{S_2}] \\ &= (\sqrt[4]{S_1} + \sqrt[4]{S_2})^2(\sqrt{S_1} + \sqrt{S_2}) . \end{aligned}$$

Solution 2. [S. En Lu] Consider an arbitrary truncated pyramid with bases A_1 and A_2 of respective areas σ_1 and σ_2, in which a sphere Γ of centre O is inscribed. Let the lateral area be σ. Suppose that C is a lateral face and that Γ touches A_1, A_2 and C in the respective points P_1, P_2 and Q.

C is a trapezoid with sides of lengths a_1 and a_2 incident with the respective bases A_1 and A_2; let h_1 and h_2 be the respective lengths of the altitudes of triangles with apexes P_1 and P_2 and bases bordering on C. By similarity (of A_1 and A_2),

$$\frac{h_1}{h_2} = \frac{a_1}{a_2} = \sqrt{\frac{\sigma_1}{\sigma_2}} \ .$$

The plane that contains these altitudes passes through P_1P_2 (a diameter of Γ) as well as Q, the point on C nearest to the centre of Γ. Since the height of C is $a_1 + a_2$ [why?], its area is

$$\begin{aligned}\frac{1}{2}(a_1+a_2)(h_1+h_2) &= \frac{1}{2}[a_1h_1 + a_2h_2 + a_1h_2 + a_2h_1] \\ &= \frac{1}{2}[a_1h_1 + a_2h_2 + 2\sqrt{a_1a_2h_1h_2}] \\ &= \frac{1}{2}[a_1h_1 + a_2h_2 + 2a_2h_2\sqrt{\sigma_1/\sigma_2}] \ .\end{aligned}$$

Adding the corresponding equations over all the lateral faces C yields

$$\sigma = \sigma_1 + \sigma_2 + \sqrt{\sigma_1\sigma_2} = (\sqrt{\sigma_1} + \sqrt{\sigma_2})^2 \ .$$

With S_0 defined as in Solution 1, we have that $S_1/S_0 = S_0/S_2$, so that $S_0 = \sqrt{S_1S_2}$. Using the results of the first paragraph applied to the truncated pyramids of bases (S_2, S_0) and (S_0, S_1), we obtain that

$$\begin{aligned}S &= (\sqrt{S_1} + \sqrt{S_0})^2 + (\sqrt{S_0} + \sqrt{S_1})^2 \\ &= (\sqrt{S_1} + \sqrt[4]{S_1S_2})^2 + (\sqrt[4]{S_1S_2} + \sqrt{S_2})^2 \\ &= (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 \ .\end{aligned}$$