

### Solutions for July-August, 2002 problems

157. Prove that if the quadratic equation  $x^2 + ax + b + 1 = 0$  has nonzero integer solutions, then  $a^2 + b^2$  is a composite integer.

*Solution.* Suppose that the integer roots are  $u$  and  $v$ . Then  $u + v = -a$  and  $uv = b + 1$ , whence

$$\begin{aligned} a^2 + b^2 &= (u + v)^2 + (uv - 1)^2 = u^2 + v^2 + u^2v^2 + 1 \\ &= (u^2 + 1)(v^2 + 1). \end{aligned}$$

Since each factor exceeds 1,  $a^2 + b^2$  is composite.

*Comment.* You should make sure that you are familiar with the relationship between the coefficients and roots of polynomials; a poor way to solve this problem is to use the quadratic formula. Note that, if, say,  $u = 0$ , then  $a^2 + b^2$  can be prime; for example, you get the value 5 when  $(u, v) = (0, 2)$ .

158. Let  $f(x)$  be a polynomial with real coefficients for which the equation  $f(x) = x$  has no real solution. Prove that the equation  $f(f(x)) = x$  has no real solution either.

*Solution 1.* Let  $g(x) = f(x) - x$ . Then,  $g(x)$  is a polynomial that never vanishes. We argue that it must always have the same sign. Suppose if possible that  $g(a) < 0 < g(b)$  for some reals  $a$  and  $b$ . Since  $g(x)$ , being a polynomial, is continuous, the Intermediate Value Theorem applies and there must be a number  $c$  between  $a$  and  $b$  for which  $g(c) = 0$ , yielding a contradiction.

Thus, either  $g(x) > 0$  for all  $x$  or else  $g(x) < 0$  for all  $x$ . Then

$$\begin{aligned} f(f(x)) - x &= f(f(x)) - f(x) + f(x) - x \\ &= g(f(x)) + g(x) \end{aligned}$$

for all real  $x$ . Since  $g$  never changes sign, both  $g(x)$  and  $g(f(x))$  have the same sign (either positive or negative) and so their sum cannot vanish. Hence  $f(f(x)) \neq x$  for any real  $x$ .

*Solution 2.* Suppose, if possible, that  $f(f(a)) = a$ . Let  $b = f(a)$ . Then  $f(b) = a$ . By hypothesis,  $b \neq a$ . Wolog, suppose that  $a < b$ . Then  $f(a) - a > 0$  and  $f(b) - b < 0$ . Since  $f(x) - x$  is a polynomial, it is continuous and so the Intermediate Value Theorem applies on the closed interval  $[a, b]$ . As it has opposite signs at the endpoints of  $[a, b]$ , it must vanish somewhere in the interior of the interval. But then this contradicts the hypothesis. Hence  $f(f(x)) = x$  can have no real solutions.

*Comment.* It is important to highlight that  $f(x)$  is a continuous function, as this is key to the result. (Can you construct a counterexample where it is not assumed that  $f$  is a polynomial or continuous?) Several of you tried to give a geometric argument for this, and apart from mangling the terminology (*e.g.*, not distinguishing between a function and its graph, points and values), operated at too intuitive a level. Notice that the statement “ $f(x) > 0$  or  $f(x) < 0$  for all  $x$ ” lacks a certain precision, as it could be interpreted to mean that at any individual point  $x$ , one of the two alternatives occurs (the function does not vanish), but that different alternative might occur at different points.

159. Let  $0 \leq a \leq 4$ . Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed  $\frac{1}{3}$ .

*Solution.* In the situation that  $0 \leq a \leq 1$ , the two curves intersect in the points  $(a/3, a/3)$  and  $(a, a)$ , and the bounded region is the triangle with these two vertices and the vertex  $(a/2, 0)$ . This triangle is contained

in the triangle with vertices  $(0, 0)$ ,  $(1/2, 0)$  and  $(1, 1)$  with area  $1/4$ . Hence, when  $0 \leq a \leq 1$ , the area of the bounded region cannot exceed  $1/4$ .

Let  $1 \leq a \leq 3$ . In this case, the bounded region is a quadrilateral with the four vertices  $(a/3, a/3)$ ,  $(a/2, 0)$ ,  $((a+2)/3, (4-a)/3)$  and  $(1, 1)$ . Noting that this quadrilateral is the result of removing two smaller triangles from a larger one (draw a diagram!), we find that its area is

$$\begin{aligned} 1 - \frac{1}{2} \cdot \frac{a}{3} \cdot \frac{a}{2} - \frac{1}{2} \cdot \frac{(4-a)}{3} \cdot \left(2 - \frac{a}{2}\right) \\ = 1 - \frac{a^2}{12} - \frac{1}{12}(4-a)^2 \\ = -\frac{a^2 - 4a + 2}{6} = \frac{1}{3} - \frac{(a-2)^2}{6}, \end{aligned}$$

whence we find that the area does not exceed  $1/3$  and is equal to  $1/3$  exactly when  $a = 2$ .

The case  $3 \leq a \leq 4$  is the symmetric image of the case  $0 \leq a \leq 1$  and we find that the area of the bounded region cannot exceed  $1/4$ .

*Comment.* This is not a difficult problem, but it does require a lot of care in keeping the situation straight and the computations sound. Always keep in mind completion of the square when it is a matter of optimizing a quadratic. Use of calculus is more cumbersome in such situations, and begs the question as to whether the optimum is a maximum or a minimum. (You can lose points for not explicitly resolving this issue.) R. Barrington Leigh solved the problem by looking at the area between the graph of  $y = 1 - |x - 1| - |2x - a|$  and the  $x$ -axis. The equation of this curve with the absolute value signs is equal to  $y = 3x - a$  for  $x \leq a/2$ , to  $y = a - x$  when  $a/2 \leq x \leq 1$  and  $a + 2 - 3x$  when  $x \geq 1$ , with suitable modification when  $a \geq 2$ . Because at each value of  $x$ , the difference in the ordinates remains the same as in the original problem, the area here is the same as the area between the two graphs of the problem. You might want to try the problem in this way.

160. Let  $I$  be the incentre of the triangle  $ABC$  and  $D$  be the point of contact of the inscribed circle with the side  $AB$ . Suppose that  $ID$  is produced outside of the triangle  $ABC$  to  $H$  so that the length  $DH$  is equal to the semi-perimeter of  $\triangle ABC$ . Prove that the quadrilateral  $AHBI$  is concyclic if and only if angle  $C$  is equal to  $90^\circ$ .

*Note.* In the solutions that follow, we use the standard notation that  $a, b, c$  are the respective lengths of the sides  $BC, CA, AB$ ,  $r$  is the inradius of the triangle and  $s$  is its semi-perimeter (so that  $2s = a + b + c$ ). We note that the area of the triangle is  $rs$ , that the distances from the vertices  $A, B, C$  to the tangent points of the incircle are respectively  $s - a, s - b, s - c$ , and that  $r = (s - c) \tan(C/2)$ . If you are not familiar with these relationships, then regard them as exercises.

*Solution 1.* The quadrilateral  $AHBI$  is concyclic if and only if  $DI : DH = DB : DA$ , or equivalently,

$$rs = (s - a)(s - b) = s^2 - s(a + b) + ab = s^2 - s(2s - c) + ab = s(c - s) + ab.$$

The area of the triangle is  $rs = \frac{1}{2}ab \sin C = abt(1 + t^2)^{-1}$  and  $r = (s - c)t$ , where  $t = \tan C/2$ . Then the concyclic condition is equivalent to

$$rs = -\frac{rs}{t} + \frac{rs(1 + t^2)}{t} = rst,$$

or  $t = 1$ . This is equivalent to  $C/2 = 45^\circ$ , and the result follows.

*Solution 2.* The concyclic condition is equivalent to

$$rs = (s - a)(s - b) \iff 4rs = (c + b - a)(c + a - b) = c^2 - b^2 - a^2 + 2ab.$$

We have, for any triangle, that  $2ab \sin C = 4rs$  (four times the area) and  $c^2 = a^2 + b^2 - 2ab \cos C$ . Hence the concyclic condition is equivalent to

$$2ab \sin C = 2ab(1 - \cos C) \iff \sin C + \cos C = 1 .$$

If  $\sin C + \cos C = 1$ , then, by squaring,  $1 + 2 \sin C \cos C = 1 \implies \sin 2C = 0$  so that  $C = 90^\circ$ . On the other hand, if  $C = 90^\circ$ , then  $\sin C + \cos C = 1$ . The result follows.

*Comment.* We can end the solution, keeping the equivalences to the end, by noting that

$$\sin C + \cos C = 1 \iff \sin(C + 45^\circ) = \sqrt{2} \iff C = 90^\circ .$$

*Solution 3.* [A. Feizmohammadi]

$$\angle AIB = 180^\circ - (\angle BAI + \angle ABI) = 180^\circ - \frac{\angle A + \angle B}{2} = 90^\circ + \frac{\angle C}{2} .$$

Also,

$$\tan \angle AHB = \frac{(s-a)/s + (s-b)/s}{1 - (s-a)(s-b)/s^2} = \frac{(2s-a-b)s}{(a+b)s - ab} = \frac{cs}{(2s-c)s - ab} = \frac{cs}{2s^2 - cs - ab} .$$

Suppose that  $\angle C$  is right. Then  $\angle AIB = 135^\circ$  and

$$c^2 = a^2 + b^2 \implies (a+b)^2 - c^2 = 2ab \implies (a+b-c)(a+b+c) = 2ab \implies ab = 2s(s-c) ,$$

so that

$$\tan \angle AHB = \frac{cs}{2s^2 - cs - 2s^2 + 2cs} = 1 \implies \angle AHB = 45^\circ .$$

Since angles  $AHB$  and  $AIB$  are supplementary,  $AHBI$  is concyclic.

On the other hand, suppose that  $AHBI$  is concyclic. Then

$$\begin{aligned} \sqrt{s(s-a)(s-b)(s-c)} &= (s-a)(s-b) = rs \implies (s-a)(s-b) = s(s-c) \\ &\implies ab = (a+b-c)s \implies 2ab = (a+b)^2 - c^2 \\ &\implies a^2 + b^2 = c^2 , \end{aligned}$$

so that  $\angle ACB$  is right.

161. Let  $a, b, c$  be positive real numbers for which  $a + b + c = 1$ . Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2} .$$

*Solution.* Observe that, by the arithmetic-geometric means inequality  $ab \leq \frac{1}{2}(a^2 + b^2)$ , so that

$$\frac{a^3}{a^2 + b^2} = a - b \frac{ab}{a^2 + b^2} \geq a - \frac{b}{2}$$

with a similar inequality for the other two terms on the left side. Adding these inequalities and using  $a + b + c = 1$  yields the desired result.

*Comment.* R. Furmaniak noted that

$$\frac{2a^3}{a^2 + b^2} - 2a + b = \frac{b(b-a)^2}{a^2 + b^2} \geq 0 ,$$

which again yields the result by concluding as in the solution.

162. Let  $A$  and  $B$  be fixed points in the plane. Find all positive integers  $k$  for which the following assertion holds:

*among all triangles  $ABC$  with  $|AC| = k|BC|$ , the one with the largest area is isosceles.*

*Solution 1.* In the case that  $k = 1$ , all triangles satisfying the condition are isosceles with  $AC = BC$ , the locus of  $C$  is the right bisector of  $AB$  and there is no triangle with largest area (the area can be arbitrarily large). Hence  $k \geq 2$ . Since, by the triangle inequality,

$$|AB| + |BC| > |AC| = k|BC| \geq 2|BC| ,$$

we must have that  $|AB| > |BC|$ . Therefore, the only way a triangle satisfying the condition can be isosceles is for  $|AB| = |AC| = k|BC|$ . Since  $|BC| < |AC|$ ,  $\angle CAB$  cannot exceed  $60^\circ$ , so that when  $k \geq 2$ , there is no isosceles triangle in the set.

*Solution 2.* As in Solution 1, we need consider only the case that  $k \geq 1$ , and we can dispose of the case  $k = 1$ . Let  $D$  be the foot of the perpendicular from  $C$  to  $AB$  (possibly produced; note that  $D$  and  $B$  lie on the same side of  $A$ ) and let the respective lengths of  $AB$ ,  $AD$  and  $CD$  be  $c$ ,  $x$  and  $v$ . We wish to maximize  $v$ . The given condition implies that

$$x^2 + v^2 = k((x - c)^2 + v^2)$$

so that

$$v^2 = -x^2 + \frac{2k^2c}{k^2 - 1}x - \frac{k^2c^2}{k^2 - 1} .$$

The maximum value of  $v$  is equal to

$$\frac{kc}{k^2 - 1}$$

assumed when  $x = \frac{k^2c}{k^2 - 1} > c$ . If this maximum value of  $v$  is assumed when  $|AC| = |AB|$ , then we have  $x^2 + v^2 = c^2$  which leads to  $k = 1/\sqrt{3}$ . If the maximum is assumed when  $|BC| = |AB|$ , then  $(x - c)^2 + v^2 = c^2$ , which leads to  $k = \sqrt{3}$ . As both solutions are extraneous, there is no positive integer value of  $k$  for which the area is maximized when the triangle is isosceles.

*Solution 3.* As before, we can deal with the  $k = 1$  case. Let  $k \neq 1$ . Let the sides of the triangle be  $(a, ka, c)$  where  $c$  is the length of  $AB$ ; let  $C$  be the angle opposite  $c$ . Then

$$c^2 = (k^2 + 1)a^2 - 2ka^2 \cos C$$

whence  $a^2 = c^2(k^2 + 1 - 2k \cos C)^{-1}$ . The area of the triangle is equal to

$$\frac{1}{2}ka^2 \sin C = \frac{1}{2}kc^2(k^2 + 1 - 2k \cos C)^{-1} \sin C .$$

The derivative of  $\sin C(k^2 + 1 - 2k \cos C)^{-1}$  with respect to  $C$  is equal to  $[(\cos C)(k^2 + 1) - 2k][k^2 + 1 - 2k \cos C]^{-2}$ , from which we can read off that area is maximized when  $\cos C = 2k(k^2 + 1)^{-1}$ . At this maximum value,  $(k^2 + 1)c^2 = (k^2 - 1)^2a^2$ . The triangle maximizing the area can be isosceles in two ways. The case  $c = a$  occurs when  $k = \sqrt{3}$ , and the case  $c = ka$  occurs when  $k = 1/\sqrt{3}$ . Neither of these is an integer.

Alternatively, the case  $c = a$  corresponds to  $\cos C = k/2$ , and equating  $k/2$  and  $2k/(k^2 + 1)$  leads to  $k^2 = 3$ . The case  $c = ka$  corresponds to  $\cos C = 1/2k$ , and this leads to  $k^2 = 1/3$ . We conclude as before.

*Solution 4.* Dispose of the  $k = 1$  case as before. Let  $k \neq 1$ . Let  $c = |AB|$  be fixed, and let the other two sides have length  $a$  and  $la$ . Sixteen times the square of the area of the triangle is, by Heron's rule,

$$\begin{aligned} & [(1+k)a + c][(1+k)a - c][c + (1-k)a][c - (1-k)a] \\ &= [(1+k)^2 a^2 - c^2][c^2 - (1-k)^2 a^2] = 2a^2 c^2 (1+k^2) - (k^2-1)^2 a^4 - c^4 \\ &= \left[ \frac{(k^2+1)^2}{(k^2-1)^2} c^4 - c^4 \right] - \left[ (k^2-1)a^2 - \frac{(k^2+1)}{(k^2-1)} c^2 \right]^2 \\ &= \frac{4k^2 c^4}{(k^2-1)^2} - (k^2-1) \left[ a^2 - \frac{k^2+1}{(k^2-1)^2} c^2 \right]. \end{aligned}$$

The maximum area occurs when  $(k^2-1)^2 a^2 = (k^2+1)c^2$ . This occurs when  $a = c$  exactly when  $(k^2-1)^2 = k^2+1$ , so  $k^2 = 3$ , and occurs when  $ka = c$  exactly when  $(k^2-1)^2 = k^4+k^2$ , so  $3k^2 = 1$ . We conclude as before.

*Comments.* Several students committed the fallacy of setting, for example,  $AC = AB$ , noting that the locus of  $C$  in this case was a circle with diameter containing  $AB$ , claiming that the area was maximized when  $\angle C = 90^\circ$  and so  $BC$  was equal to  $\sqrt{2}AC$ . They deduced that  $\sqrt{2}$  and  $1/\sqrt{2}$  were the only values of  $k$  that allowed the area to be maximized by an isosceles triangle. However, this puts things the wrong way around. The value  $k$  is fixed, and then we look at all appropriate triangles. All the solvers did was to maximize the area over all isosceles triangles for which  $AC = AB$ ; as  $C$  traces out the set of points satisfying these conditions, of course, the ratio of the lengths of  $AC$  and  $BC$  vary. Those solvers who made this mistake are invited to investigate the case  $k = \sqrt{2}$  in more detail, and this will help them understand where they went wrong.

If we coordinatize the situation with  $A \sim (0,0)$  and  $B \sim (1,0)$ , we see that the locus of  $C$  is given by

$$(k^2-1)(x^2+y^2) - 2k^2x + k^2 = 0.$$

This is the equation of a circle (of Apollonius) with centre at  $(k^2/(k^2-1), 0)$  and with radius  $k(k^2-1)^2$ . Thus, the area is maximized when  $C$  is located at the point  $(k^2(k^2-1)^{-1}, k(k^2-1)^{-1})$ . Checking when this yields an isosceles triangle leads to the two values of  $k$  that we have already seen:  $\sqrt{3}$  and  $1/\sqrt{3}$ .

163. Let  $R_i$  and  $r_i$  be the respective circumradius and inradius of triangle  $A_i B_i C_i$  ( $i = 1, 2$ ). Prove that, if  $\angle C_1 = \angle C_2$  and  $R_1 r_2 = r_1 R_2$ , then the two triangles are similar.

*Solution.* Let  $\Gamma$  be the circumcircle of  $\triangle A_1 B_1 C_1$ . Scale  $\triangle A_2 B_2 C_2$  so that  $A_2 = A_1 = A$ ,  $B_2 = B_1 = A$  and  $C_2$  and  $C_1$  lie on the same side of  $AB$ . Then  $\Gamma$  is the circumcircle of  $ABC_2$ , so that  $R_1 = R_2$  and  $r_1 = r_2$ . Using the conventional notation, we have that

$$(s_1 - c_1) \cot(C_1/2) = r_1 = r_2 = (s_2 - c_2) \cot(C_2/2)$$

whence, as  $c_1 = c_2$  and  $C_1 = C_2$ ,  $s_1 = s_2$ . Therefore  $a_1 + b_1 = a_2 + b_2$ . Since  $r_1 s_1 = r_2 s_2$ , the two triangles have the same area, so that  $a_1 b_1 \sin C_1 = a_2 b_2 \sin C_2$  and thus  $a_1 b_1 = a_2 b_2$ . Since the pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  have the same sum and product, they are roots of the same quadratic equation, and so we must have that  $(a_1, b_1) = (a_2, b_2)$  or  $(a_1, b_1) = (b_2, a_2)$ . In either case, the triangles are congruent, so that triangle  $A_1 B_1 C_1$  is a scaled version of the original triangle  $A_2 B_2 C_2$ . The result follows.

*Comment.* There were many variations using various trigonometric identities.

164. Let  $n$  be a positive integer and  $X$  a set with  $n$  distinct elements. Suppose that there are  $k$  distinct subsets of  $X$  for which the union of any four contains no more than  $n-2$  elements. Prove that  $k \leq 2^{n-2}$ .

*Solution 1.* [R. Furmaniak] We may assume that  $n \geq 2$ . We give a proof by contradiction. Suppose the  $S$  is a family of  $k$  subsets of  $X$ , where  $k > 2^{n-2}$ . Let  $X = \{x_1, x_2, \dots, x_n\}$  where the first element is selected

so that  $x_1$  belongs to some member of  $S$  but not to all. (Why can you do this?) For each  $i$ ,  $1 \leq i \leq n-1$ , let  $A_i$  be the class of all subsets of  $\{x_1, x_2, \dots, x_i\}$  which are contained in some subset of  $S$ , and  $B_i$  be the class of all subsets of  $\{x_{i+1}, \dots, x_n\}$  which are contained in some subset of  $S$ .

Note that  $2^{n-2} < k \leq |A_i||B_i|$ , since each set of  $S$  is the union of a set in  $A_i$  and a set in  $B_i$ . ( $|\cdot|$  refers to the number of elements.) Therefore, either  $|A_i| > 2^{i-1}$  or  $|B_i| > 2^{n-i-1}$ . In the former case,  $A_i$  must contain a complementary pair of subsets (use the pigeonhole principle on complementary pairs of sets) of  $\{x_1, \dots, x_i\}$ ; the latter,  $B_i$  must contain a complementary pair of sets of  $\{x_{i+1}, \dots, x_n\}$ .

By our choice of  $x_1$ , we see that  $A_1 = \{\emptyset, x_1\}$ , so that  $|A_1| = 2$ . Therefore there are values of the index  $i$  for which  $|A_i| > 2^{i-1}$ . If it turns out that  $|A_{n-1}| > 2^{n-2}$ , then  $A_{n-1}$  contains two sets whose union is  $\{x_1, \dots, x_{n-1}\}$  and so  $S$  contains two sets whose union has at least  $n-1$  elements. In this case,  $S$  fails to satisfy the hypotheses of the problem.

Suppose that  $|A_{n-1}| \leq 2^{n-2}$ . Then select the smallest index  $k$  for which  $|A_k| \leq 2^{k-1}$ . Then  $k \geq 2$  and  $|A_{k-1}| > 2^{k-2}$  while  $|B_k| > 2^{n-k-1}$ . Therefore,  $A_{k-1}$  contains two sets whose union is  $\{x_1, x_2, \dots, x_{k-1}\}$  and  $B_k$  contains two sets whose union is  $\{x_{k+1}, \dots, x_n\}$ . We conclude that  $S$  must contain four sets whose union contains the  $n-1$  elements of  $X$  apart from  $x_k$ , and so  $S$  again fails to satisfy the hypothesis of the problem.

We conclude that any family of more than  $2^{n-2}$  sets fails to satisfy the hypothesis of the problem, and so that every family that does satisfy the hypothesis must have at more  $2^{n-2}$  elements.

*Solution 2.* Let  $P$  be a class of  $k$  subsets for which the union of no four contains more than  $n-2$  elements. Suppose that  $A_1$  and  $A_2$  are two members for which the cardinality  $m = |A_1 \cup A_2|$  of the union of a pair is maximum. Since  $m < n$ , we can construct a set  $Y = A_1 \cup A_2 \cup \{y\}$ , where  $y \notin A_1 \cup A_2$ ; thus  $|Y| = m+1$ .

Let  $Q = \{Y \cap A : A \in P\}$ . No two subsets in  $Q$  are complementary (*i.e.* have union equal to  $Y$ ), so that  $|Q| \leq 2^m$ . Let  $Z = X \setminus Y$  and  $R = \{Z \cap A : A \in P\}$ . Then  $|Z| = n - (m+1) = n - m - 1$ . We prove that  $R$  has no two sets that are complementary, *i.e.*, whose union is  $Z$ . For, otherwise, suppose that  $(A_3 \cap Z) \cup (A_4 \cap Z) = Z$ , for two sets  $A_3$  and  $A_4$  in  $P$ . Then  $Z = (A_3 \cup A_4) \cap Z$ , so that

$$Z \subseteq A_3 \cup A_4 \implies A_1 \cup A_2 \cup A_3 \cup A_4 \supseteq (Y \cup Z) \setminus \{y\}$$

, whence we see that  $|A_1 \cup A_2 \cup A_3 \cup A_4| = n-1$ , contrary to hypothesis. Hence  $|R| \leq 2^{n-m-2}$ .

Since each set in  $P$  is uniquely determined by its intersections with  $Y$  and  $Z$ , we have that

$$k = |P| \leq |Q||R| \leq 2^m 2^{n-m-2} = 2^{n-2},$$

as desired.

*Comment.* Several solvers pointed out that if you took all the subsets of  $n-2$  of the elements of  $X$ , then we got an “extreme” case of a family of sets that satisfied the hypothesis and the conclusion, and essentially said that if we tried to stuff in any more sets, we would contradict the hypothesis. However, this begs the question as to whether you could drop some of these sets and replace them by a larger number of sets while still keeping the hypothesis.

165. Let  $n$  be a positive integer. Determine all  $n$ -tuples  $\{a_1, a_2, \dots, a_n\}$  of positive integers for which  $a_1 + a_2 + \dots + a_n = 2n$  and there is no subset of them whose sum is equal to  $n$ .

*Solution 1.* Let  $s_k = a_1 + a_2 + \dots + a_k$  for  $1 \leq k \leq n$ . By hypothesis, all of these are incongruent modulo  $n$ . Now let  $t_1 = a_2$ ,  $t_k = s_k$  for  $2 \leq k \leq n$ . Again, the hypothesis forces all to be incongruent. Hence the sets  $\{s_k\}$  and  $\{t_k\}$  both constitute a complete set of residues, modulo  $n$ , so it must happen that  $s_1 \equiv t_1 \equiv a_1 \pmod{n}$ . A similar argument can be marshalled when the pair  $(a_1, a_2)$  is replaced by any other pair of elements. Hence, each of the terms in the set  $\{a_i\}$  is congruent to some number  $m \pmod{n}$ , where  $0 \leq m \leq n-1$ . Now

$$mn \leq a_1 + a_2 + \dots + a_n = 2n$$

from which either  $m = 2$ ,  $n$  is odd, and the  $n$ -tuple is  $\{2, 2, 2, \dots, 2\}$  or else  $m = 1$  and the  $n$ -tuple is  $\{1, 1, 1, \dots, 1, n + 1\}$ .

*Solution 2.* [J.Y. Jin] Wolog, suppose that  $a_1 \leq a_2 \leq \dots \leq a_n$ . Note that  $a_1$  cannot exceed 2. If  $a_1 = 2$ , then each  $a_i$  is equal to 2 and  $n$  is odd. Suppose that  $a_1 = 1$ . Then  $a_n \geq 3$ . Let  $s_k = a_1 + \dots + a_k$  for  $1 \leq k \leq n$ . Then  $s_{n-1} = 2n - a_n \leq 2n - 3$ . By the hypothesis, we see that the set of the  $s_k$  with  $1 \leq k \leq n - 1$  contains at most one member from each of the  $n - 1$  sets:

$$\{1, n + 1\}, \{2, n + 2\}, \{3, n + 3\}, \dots, \{n - 3, 2n - 3\}, \{n - 2\}, \{n - 1\}$$

so it must contain *exactly* one from each set. In particular, the values  $n - 2$  and  $n - 1$  are assumed, so that for some  $i$ ,  $s_i = n - 2$  and  $s_{i+1} = n - 1$ . Thus  $a_{i+1} = 1$ , and  $a_1 = a_2 = \dots = a_{i+1} = 1$ . But this means that  $i = n - 2$ , so that  $a_{n-1} = 1$  and  $a_n = n + 1$ .

*Comment.* Some of the solvers tried an induction argument. However, the process was faulty, since they generally started with an  $n$ -tuple and then tried to build from that an  $(n + 1)$ -tuple. However, for an induction argument to work, you need to start with a general  $(n + 1)$ -tuple that satisfies the condition, and then indicate how to reduce it to the result for a smaller set.

166. Suppose that  $f$  is a real-valued function defined on the reals for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real  $x$  and  $y$ . Prove that  $f(x) \geq 0$  for all real  $x$ .

*Solution.* Suppose  $x \neq 1$ ; let  $y = x(x - 1)^{-1}$ . Then  $xy = y + x$ , so that

$$f\left(\frac{x^2 - 2x}{1 - x}\right) = f(y - x) \geq f(x + y) - f(xy) = 0.$$

For each real  $z$ , the equation  $z = (x^2 - 2x)(1 - x)^{-1}$  is equivalent to  $x^2 + (z - 2)x - z = 0$ . This quadratic equation (in  $x$ ) has a discriminant  $z^2 + 4$  which is positive and therefore it is solvable for all real  $z$ . (Note that in no case is  $x = 1$  a solution.) Hence  $f(z) \geq 0$  for all real  $z$ .

*Comment.* O. Bormashenko noted that when  $2x = \sqrt{z^2 + 4} + 2 - z$  and  $2y = \sqrt{z^2 + 4} + 2 + z$ , then  $xy = x + y$  and  $y - x = z$ .

167. Let  $u = (\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$  and  $v = (\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$ . Prove that, for each positive integer  $n$ ,  $u^n + v^{n+1} = 0$ .

*Solution.* Observe that, if  $c = a - b$ , then  $c^3 = a^3 - b^3 - 3abc$ , so that  $c^3 + 3abc - (a^3 - b^3) = 0$ . Applying this to  $u$  and  $v$  in place of  $c$ , we obtain that

$$0 = u^3 + 3u + 4 = (u - 1)(u^2 - u + 4) = (u + 1)\left[\left(u - \frac{1}{2}\right)^2 - \frac{15}{4}\right]$$

and

$$0 = v^3 + 15v + 16 = (v + 1)(v^2 - v + 16) = (v + 1)\left[\left(v - \frac{1}{2}\right)^2 - \frac{63}{4}\right].$$

Since the expressions in square brackets are negative, we must have that  $u = v = -1$ , from which the result follows.

*Comment.* Some students observed that

$$\left(\frac{\sqrt{5} \pm 1}{2}\right)^3 = \sqrt{5} \pm 2$$

and

$$\left(\frac{\sqrt{21} \pm 1}{2}\right)^3 = \sqrt{189} \pm 8,$$

and this immediately leads to the result.

168. Determine the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ.$$

*Preliminary work.* Before getting into the solution, we will discuss how to obtain the trigonometric ratios of certain angles related to  $36^\circ$ . It is useful for you to know some of these techniques, as these angles tend to come up in problems, and to be on the safe side in a contest, you should try to include a justification for assertions that you make about these angles. Here is one way to evaluate  $t = \cos 36^\circ$ . Observe that

$$\begin{aligned} t &= -\cos 144^\circ = 1 - 2\cos^2 72^\circ \\ &= 1 - 2(2t^2 - 1)^2 = -8t^4 + 8t^2 - 1 \end{aligned}$$

from which we see that

$$0 = 8t^4 - 8t^2 + t + 1 = (2t - 1)(t + 1)(4t^2 - 2t - 1).$$

Since  $t$  is equal to neither 1 nor  $\frac{1}{2}$ , we must have that  $4t^2 = 2t + 1$ . Solving this equation will give you an actual numerical value (can you justify your choice of root?).

A very useful relation is  $4 \cos 36^\circ \cos 72^\circ = 1$ . This can be checked geometrically. Let  $PQS$  be a triangle for which  $\angle P = \angle S = 36^\circ$  and  $\angle PQS = 108^\circ$ . Let  $R$  be a point on the side  $PS$  for which  $\angle PQR = 72^\circ$  and  $\angle SQR = 36^\circ$ . Then  $PQ = PR$ ,  $PQ = QS$  and  $QR = RS$ ; let  $r$  be the common length of  $PQ$ ,  $PR$ ,  $QS$  and let  $s$  be the common length of  $QR$  and  $RS$ . Then  $\cos 72^\circ = s/2r$  and  $\cos 36^\circ = r/2s$  and the desired result follows. An algebraic derivation of this result can also be given.

$$\begin{aligned} 4 \cos 36^\circ \cos 72^\circ &= \frac{4 \sin 36^\circ \cos 36^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{2 \sin 72^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{\sin 144^\circ}{\sin 36^\circ} = 1. \end{aligned}$$

We can make use of complex numbers. Let  $\zeta = \cos 72^\circ + i \sin 72^\circ$  so that  $\zeta$  is a nonreal root of

$$0 = x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1).$$

Hence  $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$ ; using de Moivre's theorem, and taking the real part of this equation, we find that

$$1 + \cos 72^\circ + \cos 144^\circ + \cos 216^\circ + \cos 288^\circ = 1 + 2 \cos 72^\circ - 2 \cos 36^\circ = 0.$$

(Note that taking the imaginary part yields a triviality.) Another way to look at this result is to note that the vectors represented by the five roots of unity sum bound a closed regular pentagon and so sum to zero.

*Solution 1.* Let  $\cos 36^\circ = t$ . Then

$$\begin{aligned} &\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &= [\cos 5^\circ + \cos 293^\circ] + [\cos 77^\circ + \cos 221^\circ] + \cos 149^\circ \\ &= \cos 149^\circ [2 \cos 144^\circ + 2 \cos 72^\circ + 1] \\ &= \cos 149^\circ [-2 \cos 36^\circ + 2 \cos 72^\circ + 1] = 0. \end{aligned}$$

Alternatively, this is seen to be equal to

$$\cos 149^\circ [-2t + 2(2t^2 - 1) + 1] = \cos 149^\circ [-2t + 4t^2 - 1] = 0 .$$

*Solution 2.* [C. Huang]

$$\begin{aligned} & \cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &= \cos 5^\circ + 2 \cos 185^\circ \cos 108^\circ + 2 \cos 185^\circ \cos 36^\circ \\ &= \cos 5^\circ [1 + 2(\cos 72^\circ - \cos 36^\circ)] = \cos 5^\circ [1 - 4 \sin 18^\circ \sin 54^\circ] \\ &= \cos 5^\circ [1 - 4 \cos 72^\circ \cos 36^\circ] = 0 . \end{aligned}$$

*Comment.* There were other solutions with similar manipulations. A quick solution can also be realized using complex numbers. The given expression is equal to the real part of

$$(\cos \theta + i \sin \theta)(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4) = 0 ,$$

where  $\zeta$  is an imaginary fifth root of unity and  $\theta$  is the angle whose degree measure is  $5^\circ$ .

169. Prove that, for each positive integer  $n$  exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 .$$

*Solution 1.* For positive integer  $n > 1$  and  $-1 < x, x \neq 0$ , we have that  $(1+x)^n > 1+nx$  (use induction). Hence, for  $n \geq 2$ ,

$$\left(1 - \frac{1}{2^n}\right)^n > 1 - \frac{n}{2^n} \geq \frac{1}{2} .$$

(The latter inequality is left as an an easy induction exercise.) Hence

$$1 - \frac{1}{2^n} > \frac{1}{2^{1/n}}$$

and the result follows.

*Solution 2.* [R. Furmaniak] By the arithmetic-geometric means inequality, we have that

$$1 - \frac{1}{2^n} = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 + \cdots + \frac{1}{2^{n-1}} \cdot 1 + \frac{1}{2^{n-1}} \cdot \frac{1}{2} > \left(\frac{1}{2}\right)^{1/(2^{n-1})} .$$

Now,  $2^{n-1} = 1 + 1 + 2 + \cdots + 2^{n-2} \leq 1 + 1 + 1 + \cdots + 1 = n$ , so that  $1/(2^{n-1}) < 1/n$  and  $(1/2)^{1/(2^{n-1})} > (1/2)^{1/n} = 1/(2^{1/n})$ . The result follows.

*Solution 3.* [S. Wong] By the arithmetic-geometric means inequality

$$\begin{aligned} \frac{1}{2^{1/n}} &= \left(\frac{1}{2} \cdot 1 \cdot 1 \cdots 1\right)^{1/n} < \frac{n-1 + \frac{1}{2}}{n} = 1 - \frac{1}{2n} \\ &\implies \frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 + \frac{1}{2^n} - \frac{1}{2n} . \end{aligned}$$

When  $n = 2$ ,  $2n = 2^n$ , while if  $n \geq 3$ ,

$$2^n = (1+1)^n = 1 + n + \cdots + \binom{n}{n-1} + 1 > 2n$$

The desired inequality follows.

170. Solve, for real  $x$ ,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4 .$$

*Solution 1.* If  $x < 0$ , the left side is negative and so there is no negative value of  $x$  which satisfies the equation. If  $x = 0$ , then the left side is undefined. Suppose that  $x > 0$ . Then by the arithmetic-geometric means inequality, we have that

$$4 = x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x \geq 2\sqrt{2^{(1/x)+x}} \geq 2\sqrt{2^2} = 4 ,$$

since  $(1/x)+x \geq 2$ . Since the outside members of this inequality are equal, we must have equality everywhere, and this occurs if and only if  $x = 1$ . Hence,  $x = 1$  is the sole solution of the equation.

*Solution 2.* As before, we see that  $x$  must be positive. The equation is equivalent to

$$\begin{aligned} 0 &= x^2 2^{1/x} - 4x + 2^x \\ &= (x \cdot 2^{1/(2x)} - 2 \cdot 2^{-1/(2x)})^2 - (4 \cdot 2^{-(1/x)} - 2^x) . \end{aligned}$$

Since the first member of the right side is nonnegative, we must have that  $4 \cdot 2^{-1/x} \geq 2^x$ , which is equivalent to  $4 \geq 2^{x+(1/x)}$  or  $x + (1/x) \leq 2$ . But the last can occur if and only if  $x = 1$ . Indeed,  $x = 1$  satisfies the equation.