Solutions

139. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 \ .$$

Solution 1. Let the tetrahedron be bounded by the three coordinate planes in $\mathbf{R}^3$ and the plane with equation $\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 1$, where u, v, w are positive. The vertices of the tetrahedron are $(0,0,0)$, $(u,0,0)$, $(0,v,0)$, $(0,0,w)$. Let d, a, b, c be the areas of the faces opposite these respective vertices. Then the volume V of the tetrahedron is equal to

$$\frac{1}{3}au = \frac{1}{3}bv = \frac{1}{3}cw = \frac{1}{3}dk \ ,$$

where k is the distance from the origin to its opposite face. The foot of the perpendicular from the origin to this face is located at $((um)^{-1}, (vm)^{-1}.(wm)^{-1})$, where $m = u^{-2} + v^{-2} + w^{-2}$, and its distance from the origin is $m^{-1/2}$. Since $a = 3Vu^{-1}$, $b = 3Vv^{-1}$, $c = 3Vw^{-1}$ and $d = 3Vm^{1/2}$, the result follows.

Solution 2. [J. Chui] Let edges of lengths x, y, z be common to the respective pairs of faces of areas (b,c), (c,a), (a,b). Then $2a = yz$, $2b = zx$ and $2c = xy$. The fourth face is bounded by sides of length $u = \sqrt{y^2+z^2}$, $v = \sqrt{z^2+x^2}$ and $w = \sqrt{x^2+y^2}$. By Heron's formula, its area d is given by the relation

$$\begin{aligned}
16d^2 &= (u+v+w)(u+v-w)(u-v+w)(-u+v+w) \\
&= [(u+v)^2 - w^2][(w^2 - (u-v)^2] = [2uv + (u^2+v^2-w^2)][2uv - (u^2+v^2-w^2)] \\
&= 2u^2v^2 + 2v^2w^2 + 2w^2u^2 - u^4 - v^4 - w^4 \\
&= 2(y^2+z^2)(x^2+z^2) + 2(x^2+z^2)(x^2+y^2) + 2(x^2+y^2)(x^2+z^2) \\
&\qquad - (y^2+z^2)^2 - (x^2+z^2)^2 - (x^2+y^2)^2 \\
&= 4x^2y^2 + 4x^2z^2 + 4y^2z^2 = 16a^2 + 16b^2 + 16c^2 \ ,
\end{aligned}$$

whence the result follows.

Solution 3. Use the notation of Solution 2. There is a plane through the edge bounding the faces of areas a and b perpendicular to the edge bounding the faces of areas c and d. Suppose it cuts the latter faces in altitudes of respective lengths u and v. Then $2c = u\sqrt{x^2+y^2}$, whence $u^2(x^2+y^2) = x^2y^2$. Hence

$$v^2 = z^2 + u^2 = \frac{x^2y^2 + x^2z^2 + y^2z^2}{x^2+y^2} = \frac{4(a^2+b^2+c^2)}{x^2+y^2} \ ,$$

so that

$$2d = v\sqrt{x^2+y^2} \Longrightarrow 4d^2 = 4(a^2+b^2+c^2) \ ,$$

as desired.

Solution 4. [R. Ziman] Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors orthogonal to the respective faces of areas a, b, c, d that point inwards from these faces and have respective magnitudes a, b, c, d. If the vertices opposite the respective faces are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{O}$, then the first three are pairwise orthogonal and $2\mathbf{c} = \mathbf{x}\times\mathbf{y}$, $2\mathbf{b} = \mathbf{z}\times\,\mathbf{x}$, $2\mathbf{c} = \mathbf{x}\times\mathbf{y}$, and $2\mathbf{d} = (\mathbf{z} - \mathbf{y})\times(\mathbf{z} - \mathbf{x}) = -\,(\mathbf{z}\times\mathbf{x}) - (\mathbf{y}\times\mathbf{z}) - (\mathbf{x}\times\mathbf{y})$. Hence $\mathbf{d} = -\,(\mathbf{a} + \mathbf{b} + \mathbf{c})$, so that

$$d^2 = \mathbf{d}\cdot\mathbf{d} = (\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c}) = a^2 + b^2 + c^2 \ .$$

140. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line

was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

Answer. When $x = 5/6$, he could have $1/7$ of a line of 42 behind him, $1/8$ of a line of 24 behind him and $1/9$ of a line of 18 behind him. When $x = 11/12$, he could have $1/14$ of a line of 84 behind him, $1/15$ of a line of 60 behind him and $1/16$ of a line of 48 behind him. When $x = 13/15$, he could have $1/8$ of a line of 120 behind him, $1/9$ of a line of 45 behind him and $1/10$ of a line of 30 behind him.

Solution 1. The strategy in this solution is to try to narrow down the search by considering a special case. Suppose that $x = (u-1)/u$ for some positive integer exceeding 1. Let $1/(u+p)$ be the fraction of the line behind Angus. Then Angus himself represents this fraction of the line:

$$1 - \left(\frac{u-1}{u} + \frac{1}{u+p}\right) = \frac{p}{u(u+p)} ,$$

so that there would be $u(u+p)/p$ people in line. To make this an integer, we can arrange that u is a multiple of p. For $n = u+1$, we want to get an integer for $p = 1, 2, 3$, and so we may take u to be any multiple of 6. Thus, we can arrange that x is any of $5/6$, $11/12$, $17/18$, $23/24$, and so on.

More generally, for $u(u+1)$, $u(u+2)/2$ and $u(u+3)/3$ to all be integers we require that u be a multiple of 6, and so can take $n = 6k+1$. On Monday, there would be $36k^2 + 6k$ people in line with $36k^2 - 1$ in front and $6k$ behind; on Tuesday, $18k^2 + 6k$ with $18k^2 + 3k - 1$ in front and $3k$ behind; on Wednesday, $12k^2 + 6k$ with $12k^2 + 4k - 1$ and $2k$ behind.

Solution 2. [O. Bormashenko] On the three successive days, the total numbers numbers of people in line are un, $v(n+1)$ and $w(n+2)$ for some positive integers u, v and w. The fraction of the line constituted by Angus and those behind him is

$$\frac{1}{un} + \frac{1}{n} = \frac{1}{v(n+1)} + \frac{1}{n+1} = \frac{1}{w(n+2)} + \frac{1}{n+2} .$$

These yield the equations

$$(n-v)(n+1+u) = n(n+1)$$

and

$$(n+1-w)(n+2+v) = (n+1)(n+2) .$$

We need to find an integer v for which $n-v$ divides $n(n+1)$ and $n+2+v$ divides $(n+1)(n+2)$. This is equivalent to determining p, q for which $p+q = 2(n+1)$, $p < n$, p divides $n(n+1)$, $q > n+2$ and q divides $(n+1)(n+2)$. The triple $(n,p,q) = (7,4,12)$ works and yields $(u,v,w) = (6,3,2)$. In this case, $x = 5/6$.

Comment 1. Solution 1 indicates how we can select x for which the amount of the line behind Angus is represented by any number of consecutive integer reciprocals. For example, in the case of $x = 11/12$, he could also have $1/13$ of a line of 156 behind him. Another strategy might be to look at $x = (u-2)/u$, *i.e.* successively at $x = 3/5, 5/7, 7/9, \cdots$. In this case, we assume that $1/(u-p)$ is the line is behind him, and need to ensure that $u - 2p$ is a positive divisor of $u(u-p)$ for three consecutive values of p. If u is odd, we can achieve this with u any odd multiple of 15, starting with $p = \frac{1}{2}(u-1)$.

Comment 2. With the same fraction in front on two days, suppose that $1/n$ of a line of u people is behind the man on the first day, and $1/(n+1)$ of a line of v people is behind him on the second day. Then

$$\frac{1}{u} + \frac{1}{n} = \frac{1}{v} + \frac{1}{n+1}$$

so that $uv = n(n+1)(u-v)$. This yields both $(n^2+n-v)u = (n^2+n)v$ and $(n^2+n+u)v = (n^2+n)u$, leading to

$$u - v = \frac{u^2}{n^2+n+u} = \frac{v^2}{n^2+n-v} .$$

Two immediate possibilities are $(n, u, v) = (n, n+1, n)$ and $(n, u, v) = (n, n(n+1), \frac{1}{2}n(n+1))$. To get some more, taking $u - v = k$, we get the quadratic equation

$$u^2 - ku - k(n^2 + n) = 0$$

with discriminant

$$\Delta = k^2 + 4(n^2 + n)k = [k + 2(n^2 + n)]^2 - 4(n^2 + n)^2 \ ,$$

a pythagorean relationship when Δ is square and the equation has integer solutions. Select α, β, γ so that $\gamma\alpha\beta = n^2 + n$ and let $k = \gamma(\alpha^2 + \beta^2 - 2\alpha\beta) = \gamma(\alpha - \beta)^2$; this will make the discriminant Δ equal to a square.

Taking $n = 3$, for example, yields the possibilities $(u, v) = (132, 11)$, (60, 10), (36, 9), (24, 8), (12, 6), (6, 4),(4, 3). In general, we find that $(n, u, v) = (n, \gamma\alpha(\alpha - \beta), \gamma\beta(\alpha - \beta))$ when $n^2 + n = \gamma\alpha\beta$ with $\alpha > \beta$. It turns out that $k = u - v = \gamma(\alpha - \beta)^2$.

141. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n + 1)/n$, where n is a positive integer?

Solution 1. We begin by proving a more general result. Let m be a positive integer, and denote by $d(m)$ and $d(m + 1)$, the number of positive divisors of m and $m + 1$ respectively. Suppose that

$$\frac{m+1}{m} = \frac{p+1}{p} \cdot \frac{q+1}{q} \ ,$$

where p and q are positive integers exceeding m. Then $(m + 1)pq = m(p + 1)(q + 1)$, which reduces to $(p - m)(q - m) = m(m + 1)$. It follows that $p = m + u$ and $q = m + v$, where $uv = m(m + 1)$. Hence, every representation of $(m + 1)/m$ corresponds to a factorization of $m(m + 1)$.

On the other hand, observe that, if $uv = m(m + 1)$, then

$$\begin{aligned}\frac{m+u+1}{m+u} \cdot \frac{m+v+1}{m+v} &= \frac{m^2 + m(u+v+2) + uv + (u+v) + 1}{m^2 + m(u+v) + uv} \\ &= \frac{m^2 + (m+1)(u+v) + m(m+1) + 2m + 1}{m^2 + m(u+v) + m(m+1)} \\ &= \frac{(m+1)^2 + (m+1)(u+v) + m(m+1)}{m^2 + m(u+v) + m(m+1)} \\ &= \frac{(m+1)[(m+1) + (u+v) + m]}{m[m + (u+v) + m + 1]} = \frac{m+1}{m} \ .\end{aligned}$$

Hence, there is a one-one correspondence between representations and pairs (u, v) of complementary factors of $m(m + 1)$. Since m and $m + 1$ are coprime, the number of factors of $m(m + 1)$ is equal to $d(m)d(m + 1)$, and so the number of representations is equal to $\frac{1}{2}d(m)d(m + 1)$.

Now consider the case that $m = 2001$. Since $2001 = 3 \times 23 \times 29$, $d(2001) = 8$; since $2002 = 2 \times 7 \times 11 \times 13$, $d(2002) = 16$. Hence, the desired number of representations is 64.

Solution 2. [R. Ziman] Let m be an arbitrary positive integer. Then, since $(m+1)/m$ is in lowest terms, pq must be a multiple of m. Let $m + 1 = uv$ for some positive integers u and v and $m = rs$ for some positive integers r and s, where r is the greatest common divisor of m and p; suppose that $p = br$ and $q = as$, with s being the greatest common divisor of m and q. Then, the representation must have the form

$$\frac{m+1}{m} = \frac{au}{br} \cdot \frac{bv}{as} \ ,$$

where $au = br + 1$ and $bv = as + 1$. Hence

$$bv = \frac{br + 1}{u} s + 1 = \frac{brs + s + u}{u} \ ,$$

so that $b = b(uv - rs) = s + u$ and

$$a = \frac{sr + ur + 1}{u} = \frac{m + 1 - ur}{u} = v + r \ .$$

Thus, a and b are uniquely determined. Note that we can get a representation for any pair (u, v) of complementary factors or $m + 1$ and (r, s) of complementary factors of m, and there are $d(m + 1)d(m)$ of selecting these. However, the selections $\{(u, v), (r, s)\}$ and $\{(v, u), (s, r)\}$ yield the same representation, so that number of representations is $\frac{1}{2} d(m + 1)d(m)$. The desired answer can now be found.

142. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

Solution 1. [R. Barrington Leigh] We have that

$$x - y \geq x^3 + y^3 > x^3 - y^3 \ .$$

Since $x - y \geq x^3 + y^3 > 0$, we can divide this inequality by $x - y$ to obtain

$$1 > x^2 + xy + y^2 > x^2 + y^2 \ .$$

Solution 2. [S.E. Lu]

$$\begin{aligned} x - y \geq x^3 + y^3 > x^3 &> x^3 - [y^3 + xy(x - y)] \\ &= x^3 - x^2 y + xy^2 - y^3 = (x^2 + y^2)(x - y) \ , \end{aligned}$$

whereupon a division by the positive quantity $x - y$ yields that $1 > x^2 + y^2$.

Solution 3. [O. Bormashenko] Observe that $y < x$ and that $x^3 < x^3 + y^3 \leq x - y < x$, so that $0 < y < x < 1$. It follows that

$$x(x + y) < 2 \Longrightarrow xy(x + y) < 2xy \ . \qquad (1)$$

The given condition can be rewritten

$$(x + y)(x^2 + y^2) - xy(x + y) \leq x - y \ . \qquad (2)$$

Adding inequalities (1) and (2) yields

$$(x + y)(x^2 + y^2) < x + y \ ,$$

whence $x^2 + y^2 < 1$.

Solution 4. [R. Furmaniak] We have that

$$\begin{aligned} (x - y)(1 - x^2 - y^2) &= (x - y) - (x^3 - x^2 y + xy^2 - y^3) \\ &\geq (x^3 + y^3) - (x^3 - x^2 y + xy^2 - y^3) = 2y^3 + x^2 y - xy^2 = y(x^2 - xy + 2y^2) \\ &= y[(x - \sqrt{2} y)^2 + (2\sqrt{2} - 1)xy] \geq 0 \ , \end{aligned}$$

from which the result follows upon division by $x - y$.

Solution 5. Let $y = tx$. Since $x > y > 0$, we have that $0 < t < 1$. Then $x^3(1+t^3) \leq x(1-t) \Rightarrow x^2(1+t^3) \leq (1-t)$. Therefore,

$$\begin{aligned} x^2 + y^2 = x^2(1+t^2) &\leq \left(\frac{1-t}{1+t^3}\right)(1+t^2) \\ &= \frac{1-t+t^2-t^3}{1+t^3} = 1 - \frac{t(1-t+2t^2)}{1+t^3} \ . \end{aligned}$$

Since $1 - t + 2t^2$, having negative discriminant, is always positive, the desired result follows.

Solution 6. [J. Chui] Suppose, if possible, that $x^2+y^2 = r^2 > 1$. We can write $x = r\sin\theta$ and $y = r\cos\theta$ for $0 \leq \theta \leq \pi/2$. Then

$$\begin{aligned} x^3 + y^3 - (x-y) &= r^3\sin^3\theta + r^3\cos^3\theta - r\sin\theta + r\cos\theta \\ &> r\sin\theta(\sin^2\theta - 1) + r\cos^3\theta + r\cos\theta \\ &= -r\sin\theta\cos^2\theta + r\cos^3\theta + r\cos\theta \\ &= r\cos^2\theta\left(\cos\theta + \frac{1}{\cos\theta} - \sin\theta\right) \\ &> r\cos^2\theta(2-\sin\theta) > 0 \ , \end{aligned}$$

contrary to hypothesis. The result follows by contradiction.

Solution 7. Let $r > 0$ and $r^2 = x^2 + y^2$. Since $x > y > 0$, we can write $x = r\cos\theta$ and $y = r\sin\theta$, where $0 < \theta < \pi/4$. The given equality is equivalent to

$$r^2 \leq \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} \ ,$$

so it suffices to show that the right side does not exceed 1 to obtain the desired $r^2 \leq 1$.

Observe that

$$\begin{aligned} 1 - \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} &= \frac{(\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta) - (\cos\theta - \sin\theta)}{\cos^3\theta + \sin^3\theta} \\ &= \frac{\sin\theta(2 - \cos\theta\sin\theta - \cos^2\theta)}{\cos^3\theta + \sin^3\theta} > 0 \ , \end{aligned}$$

from which the desired result follows.

Solution 8. Begin as in Solution 7. Then

$$\begin{aligned} \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} &= \frac{\cos^2\theta - \sin^2\theta}{(\cos\theta + \sin\theta)^2(1 - \cos\theta\sin\theta)} \\ &= \frac{\cos 2\theta}{(1+\sin 2\theta)(1 - \frac{1}{2}\sin\theta})} = \frac{\cos 2\theta}{1 + \frac{1}{2}\sin 2\theta(1 - \sin 2\theta)} < 1 \ , \end{aligned}$$

from which the result follows.

143. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001\cdots$. What is the 2002th term of the sequence?

Solution. Let us define finite sequences as follows. Suppose that $S_1 = 0$. Then, for each $k \geq 2$, S_k is obtained by replacing each 0 in S_{k-1} by 01 and each 1 in S_{k-1} by 001. Thus,

$$S_1 = 0; \quad S_2 = 01; \quad S_3 = 01001; \quad S_4 = 010010101001; \quad S_5 = 01001010100101001010010101001; \cdots$$

Each S_{k-1} is a prefix of S_k; in fact, it can be shown that, for each $k \geq 3$,

$$S_k = S_{k-1} * S_{k-2} * S_{k-1} \ ,$$

where $*$ indicates juxtaposition. The respective number of symbols in S_k for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ is equal to 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378.

The 2002th entry in the given infinite sequence is equal to the 2002th entry in S_{10}, which is equal to the $(2002 - 985 - 408)$th $= (609)$th entry in S_9. This in turn is equal to the $(609 - 408 - 169)$th $= (32)$th entry in S_8, which is equal to the (32)th entry of S_6, or the third entry of S_3. Hence, the desired entry is 0.

Comment. Suppose that $f(n)$ is the position of the nth one, so that $f(1) = 2$ and $f(2) = 5$. Let $g(n)$ be the number of zeros up to and including the nth position, and so $n - g(n)$ is the number of ones up to and including the nth position. Then we get the two equations

$$f(n) = 2g(n) + 3(n - g(n)) = 3n - g(n) \tag{1}$$

$$g(f(n)) = f(n) - n \ . \tag{2}$$

These two can be used to determine the positions of the ones by stepping up; for example, we have $f(2) = 5$, $g(5) = 3$, $f(5) = 15 - 3 = 12$, $g(12) = f(5) - 5 = 7$, and so on. By messing around, one can arrive at the result, but it would be nice to formulate this approach in a nice clean efficient zeroing in on the answer.

144. Let a, b, c, d be rational numbers for which $bc \neq ad$. Prove that there are infinitely many rational values of x for which $\sqrt{(a + bx)(c + dx)}$ is rational. Explain the situation when $bc = ad$.

Solution 1. We study the possibility of making $c + dx = (a + bx)t^2$ for some rational numbers t. This would require that

$$x = \frac{c - at^2}{bt^2 - d} \ .$$

Since the condition $bc \neq ad$ prohibits $b = d = 0$, at least one of b and d must fail to vanish. Let us now construct our solution.

Let t be an arbitrary positive rational number for which $bt^2 \neq d$. Then $a + bx = (bc - ad)(bt^2 - d)^{-1}$ and $c + dx = (bc - ad)t^2(bt^2 - d)^{-1}$, whence

$$\sqrt{(a + bx)(c + dx)} = |(bc - ad)t(bt^2 - d)^{-1}|$$

is rational.

We need to show that distinct values of t deliver distinct values of x. Let u and v be two values of t for which

$$\frac{c - au^2}{bu^2 - d} = \frac{c - av^2}{bv^2 - d} \ .$$

Then

$$\begin{aligned} 0 &= (c - au^2)(bv^2 - d) - (c - av^2)(bu^2 - d) \\ &= bc(v^2 - u^2) + ad(u^2 - v^2) = (bc - ad)(u^2 - v^2) \ , \end{aligned}$$

so that $u = v$, and the result follows.

Consider the case that $bc = ad$. if both sides equal zero, then one of the possibilities $(a, b, c, d) = (0, 0, c, d), (a, b, 0, 0), (a, 0, c, 0), (0, b, 0, d)$ must hold. In the first two cases, any x will serve. In the third, any value of x will serve provided that ac is a rational square, and in the fourth, provided bd is a rational square; otherwise, no x can be found. Otherwise, let $c/a = d/b = s$, for some nonzero rational s, so that $(a + bx)(c + dx) = s(a + bx)^2$. If s is a rational square, any value of x will do; if s is irrational, then only $x = -a/b = -c/d$ will work.

Solution 2. $(a + bx)(c + dx) = r^2$ for rational r is equivalent to

$$bdx^2 + (ad + bc)x + (ac - r^2) = 0 \ .$$

If $b = d = 0$, this is satisfiable by all rational x provided ac is a rational square and $r^2 = ac$, and by no rational x otherwise. If exactly one of b and d is zero and $ad + bc \neq 0$, then each positive rational value is assumed by $\sqrt{(a + bx)(c + dx)}$ for a suitable value of x.

Otherwise, let $bd \neq 0$. Then, given r, we have the corresponding

$$x = \frac{-(ad + bc) \pm \sqrt{(ad - bc)^2 + 4bdr^2}}{2bd} \ .$$

If $ad = bc$, then this yields a rational x if and only if bd is a rational square. Let $ad \neq bc$. We wish to make $(ad - bc)^2 + 4bdr^2 = s^2$ for some rational s. This is equivalent to

$$4bdr^2 = (ad - bc)^2 - s^2 = (ad - bc + s)(ad - bc - s) \ .$$

Pick a pair u, v of rationals for which $u + v \neq 0$ and $uv = bd$. We want to make

$$2ur = ad - bc + s \qquad \text{and} \qquad 2vr = ad - bc - s$$

so that $(u + v)r = ad - bc$ and $s = (u - v)r$. Thus, let

$$r = \frac{ad - bc}{u + v} \ .$$

Then

$$\begin{aligned} (ad - bc)^2 + 4bdr^2 &= (ad - bc)^2 + 4uvr^2 \\ &= \left(\frac{ad - bc}{u + v}\right)^2 [(u + v)^2 - 4uv] \\ &= \frac{(ad - bc)^2(u - v)^2}{(u + v)^2} \end{aligned}$$

is a rational square, and so x is rational. There are infinitely many possible ways of choosing u, v and each gives a different sum $u + v$ and so a different value of r and x. The desired result follows.