PROBLEMS FOR SEPTEMBER

Send your solutions to Prof. E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3 no later than **October 15, 2002**. Please make sure that the front page of your solution contains your complete mailing address and your email address.

171. Let n be a positive integer. In a round-robin match, n teams compete and each pair of teams plays exactly one game. At the end of the match, the ith team has x_i wins and y_i losses. There are no ties. Prove that
$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 \ .$$

172. Let a, b, c, d. e, f be different integers. Prove that
$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-a)^2 \geq 18 \ .$$

173. Suppose that a and b are positive real numbers for which $a + b = 1$. Prove that
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} \ .$$
Determine when equality holds.

174. For which real value of x is the function
$$(1-x)^5(1+x)(1+2x)^2$$
maximum? Determine its maximum value.

175. ABC is a triangle such that $AB < AC$. The point D is the midpoint of the arc with endpoints B and C of that arc of the circumcircle of ΔABC that contains A. The foot of the perpendicular from D to AC is E. Prove that $AB + AE = EC$.

176. Three noncollinear points A, M and N are given in the plane. Construct the square such that one of its vertices is the point A, and the two sides which do not contain this vertex are on the lines through M and N respectively. [*Note:* In such a problem, your solution should consist of a description of the construction (with straightedge and compasses) and a proof in correct logical order proceeding from what is given to what is desired that the construction is valid. You should deal with the feasibility of the construction.]

177. Let a_1, a_2, $\cdots$, a_n be nonnegative integers such that, whenever $1 \leq i$, $1 \leq j$, $i + j \leq n$, then
$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1 \ .$$
(a) Give an example of such a sequence which is not an arithmetic progression.

(b) Prove that there exists a real number x such that $a_k = \lfloor kx \rfloor$ for $1 \leq k \leq n$.

Since solutions are still being marked for the June set of problems, their solutions will not be published until October.