PROBLEMS FOR NOVEMBER

Please send your solution to
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It is very important that the front page contain your complete mailing address and your email address. The deadline for this set is **December 21, 2002**.

Notes: The sides of a right-angled triangle that are adjacent to the right angle are called *legs*. The *centre of gravity* or *centroid* of a collection of n mass particles is the point where the cumulative mass can be regarded as concentrated so that the motion of this point, when exposed to outside forces such as gravity, is identical to that of the whole collection. To illustrate this point, imagine that the mass particles are connected to a point by rigid non-material sticks (with mass 0) to form a structure. The point where the tip of a needle could be put so that this structure is in a state of balance is its centroid. In addition, there is an intuitive definition of a centroid of a lamina, and of a solid: The centroid of a lamina is the point, which would cause equilibrium (balance) when the tip of a needle is placed underneath to support it. Likewise, the centroid of a solid is the point, at which the solid "balances", *i.e.*, it will not revolve if force is applied. The centroid, G of a set of points is defined vectorially by

$$\overrightarrow{OG} = \frac{\sum_{i=1}^{n} m_i \cdot \overrightarrow{OM}_i}{\sum_{i=1}^{n} m_i}$$

where m_i is the mass of the particle at a position M_i (the summation extending over the whole collection). Problem 181 is related to the centroid of an assembly of three particles placed at the vertices of a given triangle. The *circumcentre* of a triangle is the centre of its circumscribed circle. The *orthocentre* of a triangle is the intersection point of its altitudes. An *unbounded* region in the plane is one not contained in the interior of any circle.

185. Find all triples of natural numbers a, b, c, such that all of the following conditions hold: (1) $a < 1974$; (2) b is less than c by 1575; (3) $a^2 + b^2 = c^2$.

186. Find all natural numbers n such that there exists a convex $n-$sided polygon whose diagonals are all of the same length.

187. Suppose that p is a real parameter and that

$$f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36 \ .$$

(a) Check that $f(3-p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of $4\sqrt{2}$.

188. (a) The measure of the angles of an acute triangle are α, β and γ degrees. Determine (as an expression of α, β, γ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the orthocentre of the triangle; (ii) the circumcentre of the triangle.

(b) The sides of an arbitrary triangle are a, b, c units in length. Determine (as an expression of a, b, c) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the centre of the inscribed circle of the triangle; (ii) the intersection point of the three segments joining the vertices of the triangle to the points on the opposite sides where the inscribed circle is tangent (be sure to prove that, indeed, the three segments intersect in a common point).

189. There are n lines in the plane, where n is an integer exceeding 2. No three of them are concurrent and no two of them are parallel. The lines divide the plane into regions; some of them are closed (they

have the form of a convex polygon); others are unbounded (their borders are broken lines consisting of segments and rays).

(a) Determine as a function of n the number of unbounded regions.

(b) Suppose that some of the regions are coloured, so that no two coloured regions have a common side (a segment or ray). Prove that the number of regions coloured in this way does not exceed $\frac{1}{3}(n^2 + n)$.

190. Find all integer values of the parameter a for which the equation

$$|2x + 1| + |x - 2| = a$$

has exactly one integer among its solutions.

191. In **Olymonland** the distances between every two cities is different. When the transportation program of the country was being developed, for each city, the closest of the other cities was chosen and a highway was built to connect them. All highways are line segments. Prove that

(a) no two highways intersect;

(b) every city is connected by a highway to no more than 5 other cities;

(c) there is no closed broken line composed of highways only.