PROBLEMS FOR JUNE

Please submit your solutions to
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no later than **July 21, 2002**.

151. Prove that, for any natural number n, the equation

$$x(x+1)(x+2)\cdots(x+2n-1)+(x+2n+1)(x+2n+2)\cdots(x+4n)=0$$

does not have real solutions.

152. Andrew and Brenda are playing the following game. Taking turns, they write in a sequence, from left to right, the numbers 0 or 1 until each of them has written 2002 numbers (to produce a 4004-digit number). Brenda is the winner if the sequence of zeros and ones, considered as a binary number (*i.e.*, written to base 2), can be written as the sum of two integer squares. Otherwise, the winner is Andrew. Prove that the second player, Brenda, can always win the game, and explain her winning strategy (*i.e.*, how she must play to ensure winning every game).

153. (a) Prove that, among any 39 consecutive natural numbers, there is one the sum of whose digits (in base 10) is divisible by 11.

(b) Present some generalizations of this problem.

154. (a) Give as neat a proof as you can that, for any natural number n, the sum of the squares of the numbers $1, 2, \cdots, n$ is equal to $n(n+1)(2n+1)/6$.

(b) Find the least natural number n exceeding 1 for which $(1^2+2^2+\cdots+n^2)/n$ is the square of a natural number.

155. Find all real numbers x that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x+\sin x)]}-2^{\log_2(\cos x-\sin x)}=\sqrt{2}\ .$$

[The logarithms are taken to bases 3 and 2 respectively.]

156. In the triangle ABC, the point M is from the inside of the angle BAC such that $\angle MAB=\angle MCA$ and $\angle MAC=\angle MBA$. Similarly, the point N is from the inside of the angle ABC such that $\angle NBA=\angle NCB$ and $\angle NBC=\angle NAB$. Also, the point P is from the inside of the angle ACB such that $\angle PCA=\angle PBC$ and $\angle PCB=\angle PAC$. (The points M, N and P each could be inside or outside of the triangle.) Prove that the lines AM, BN and CP are concurrent and that their intersection point belongs to the circumcircle of the triangle MNP.