PROBLEMS FOR JULY AND AUGUST

Because of the variability of summer plans, the usual ration of problems has been doubled and the deadline set later so that students can have a chance to organize their work conveniently. Send your solutions to Prof. E.J. Barbeau, Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3 no later than **September 10, 2002**. Please make sure that the front page of your solution contains your complete mailing address and your email address.

Notes. A *composite integer* is one that has positive divisors other than 1 and itself; it is not prime. A set of point in the plane is *concyclic* (or *cyclic, inscribable*) if and only if there is a circle that passes through all of them.

157. Prove that if the quadratic equation $x^2 + ax + b + 1 = 0$ has nonzero integer solutions, then $a^2 + b^2$ is a composite integer.

158. Let $f(x)$ be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

159. Let $0 \leq a \leq 4$. Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed $\frac{1}{3}$.

160. Let I be the incentre of the triangle ABC and D be the point of contact of the inscribed circle with the side AB. Suppose that ID is produced outside of the triangle ABC to H so that the length DH is equal to the semi-perimeter of ΔABC. Prove that the quadrilateral $AHBI$ is concyclic if and only if angle C is equal to $90°$.

161. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2} \ .$$

162. Let A and B be fixed points in the plane. Find all positive integers k for which the following assertion holds:

among all triangles ABC with $|AC| = k|BC|$, the one with the largest area is isosceles.

163. Let R_i and r_i re the respective circumradius and inradius of triangle $A_iB_iC_i$ $(i = 1, 2)$. Prove that, if $\angle C_1 = \angle C_2$ and $R_1r_2 = r_1R_2$, then the two triangles are similar.

164. Let n be a positive integer and X a set with n distinct elements. Suppose that there are k distinct subsets of X for which the union of any four contains no more that $n-2$ elements. Prove that $k \leq 2^{n-2}$.

165. Let n be a positive integer. Determine all $n-$tples $\{a_1, a_2, \cdots, a_n\}$ of positive integers for which $a_1 + a_2 + \cdots + a_n = 2n$ and there is no subset of them whose sum is equal to n.

166. Suppose that f is a real-valued function defined on the reals for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real x and y. Prove that $f(x) \geq 0$ for all real x.

167. Let $u = (\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$ and $v = (\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$. Prove that, for each positive integer n, $u^n + v^{n+1} = 0$.

168. Determine the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \ .$$

169. Prove that, for each positive integer n exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 \ .$$

170. Solve, for real x,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4 \ .$$