Solutions

85. Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0 \ .$$

Solution 1. Suppose that the system is solvable; note that $x = 0$ is not a solution. Then $\cos ax = -\cos bx$ so that $\sin ax = \epsilon \sin bx$ where $\epsilon = \pm 1$. Hence $(a + \epsilon b) \sin bx = 0$. Since a and b are positive and unequal, $a + \epsilon b \neq 0$, so that $\sin bx = 0$. Hence $bx = n\pi$ for some integer n. Also $\sin ax = 0$, so that $ax = m\pi$ for some integer m. Hence, we must have $an = bm$ for some integers m and n. Since $\cos m\pi = -\cos n\pi$, m and n must have opposite parity.

Suppose that $a = 2^u p$ and $b = 2^v q$ with u and v unequal integers and p and q odd. Then $x = 2^{-w}\pi$. where w is the minimum of u and v satisfies the system of equations.

Solution 2. First, observe that $x \neq 0$ for any solution. If the system is satisfied, then

$$\begin{aligned}
& 0 = \cos ax + \cos bx = 2 \cos \frac{1}{2}(a+b)x \cos \frac{1}{2}(a-b)x \\
& \Longrightarrow \cos \frac{1}{2}(a+b)x = 0 \quad \text{ or } \quad \cos \frac{1}{2}(a-b)x = 0 \\
& \Longrightarrow \frac{1}{2}(a+b)x = (2k+1)\frac{\pi}{2} \quad \text{ or } \quad \frac{1}{2}(a-b)x = (2k+1)\frac{\pi}{2} \\
& \Longrightarrow ax \pm bx = (2k+1)\pi \quad \text{ for some integer } \ k \\
& \Longrightarrow \sin ax = -\sin(\pm bx) = \mp \sin bx \\
& \Longrightarrow 0 = a \sin ax + b \sin bx = (a \mp b) \sin bx \ .
\end{aligned}$$

Since $a \neq b$ and $a + b > 0$, $0 = \sin bx = \sin ax$, so that $ax = m\pi$ and $bx = n\pi$ for some integers m and n. Since $0 = \cos ax + \cos bx = (-1)^m + (-1)^n$, the integers m and n must have different parity. Hence

$$x = \frac{m\pi}{a} = \frac{n\pi}{b}$$

where m and n are integers not both even or both odd. Since $x \neq 0$, $a/b = m/n$, so a/b in lowest terms must have numerator and denominator of different parities.

We now show that, for any pair a, b satisfying this condition, there is a solution. Wolog, let $a = 2uw$ and $v = (2v + 1)w$, where the greatest common divisor of $2u$ and $2v + 1$ is 1, and w is an arbitrary positive integer. Suppose that $x = \pi/w$. Then

$$\cos ax + \cos bx = \cos 2u\pi + \cos(2v+1)\pi = 1 - 1 = 0$$

and

$$a \sin ax + b \sin bx = a \sin 2u\pi + b \sin(2v+1)\pi = 0 + 0 = 0$$

as desired.

Solution 3. Since $\cos^2 ax = \cos^2 bx$ and $a^2 \sin^2 ax + b^2 \sin^2 bx$, then

$$a^2 \cos^2 bx + b^2 \sin^2 bx = a^2 \Longrightarrow (b^2 - a^2) \sin^2 bx = 0$$

so that $bx = n\pi$ for some integer. Similarly $ax = m\pi$. The solution can be completed as before.

Comment. Note that there are two parts to the solution of this problem, and your write-up should make sure that these are carefully delineated. First, assuming that there is a solution, you derive necessary conditions on a and b that the two equations are consistent. Then, you assume these conditions on a and b, and then display a solution to the two equations. A complete solution requires noting that suitable numbers a and b actually do lead to a solution.

86. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. Prove that

(a) it is possible to inscribe a circle in it;

(b) it is possible to circumscribe a circle about it if and only if $AB \perp BC$;

(c) if $AB \perp BC$ and R and r are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$.

Comment. Most students picked up the typo in part (c) in which AC was given instead of the intended BC. I am sorry for the mistake. However, this does happen from time to time even on competitions, and you should be alert. From the context of this problem, the intention was probably pretty clear (in fact, some of you might not have realized that there was an error). The rule in such a situation is that, if you feel that there is an error, make a reasonable *nontrivial* interpretation of the problem, state it clearly and solve it.

Solution 1. (a) Triangles ABC and ADC are congruent (SSS) with the congruence implemented by a reflection in AC. Hence AC bisects angles DAB and DCB. The angle bisectors of $\angle ADB$ and $\angle ABC$ are reflected images and intersect in I, a point on AC. Since I is equidistant from the four sides of the kite $ABCD$, it is the centre of its incircle.

(b) If $AB \perp BC$, then the circle with diameter AC passes through B. By symmetry about AC, it must pass through D as well. Conversely, let $\mathcal{C}$ be the circumcircle of $ABCD$. The circle goes to itself under reflection in AC, so AC must be a diameter of $\mathcal{C}$. Hence $\angle ABC = \angle ADC = 90°$.

(c) Let I be the incentre and O the circumcentre of $ABCD$; both lie on AC. Suppose that J and K are the respective feet of the perpendiculars to AB and BC from I, and P and Q the respective feet of the perpendiculars to AB and BC from O. Let $x = |AB|$ and $y = |BC|$. Then

$$\frac{x}{y} = \frac{x-r}{r} \Rightarrow xy = r(x+y)\ .$$

Since $|AO| = |OC| = R$, $4R^2 = x^2 + y^2$. Noting that x and y both exceed r, we have that

$$\begin{aligned}(x+y)^2 &= x^2 + y^2 + 2xy = 4R^2 + 2r(x+y)\\ &\Rightarrow (x+y-r)^2 = r^2 + 4R^2\\ &\Rightarrow x + y = r + \sqrt{r^2 + 4R^2}\ .\end{aligned}$$

Now

$$\begin{aligned}|IO|^2 &= |JP|^2 + |KQ|^2 = (|JB| - \frac{1}{2}|AB|)^2 + (|KB| - \frac{1}{2}|BC|)^2\\ &= 2r^2 - r(x+y) + \frac{1}{4}(x^2+y^2) = r^2 - r\sqrt{r^2+4R^2} + R^2\ ,\end{aligned}$$

yielding the desired result.

Solution 2. (a) Since triangles ADB and CDB are isosceles, the angle bisectors of A and C right bisect the base BD and so they coincide. The line AC is an axis of reflective symmetry that interchanges B and D, and also interchanges the angle bisectors of B and D. The point P where one of the bisectors intersects the axis AC is fixed by the reflection and so lies on the other bisector. Hence, P is common to all four angle bisectors, and so is equidistant from the four sides of the quadrilateral. Thus, we can inscribe a circle inside $ABCD$ with centre P.

(b) Since AC is a line of symmetry, $\angle B = \angle D$. Note that, $ABCD$ has a circumcircle $\Leftrightarrow$ pairs of opposite angles sum to $180°$ $\Leftrightarrow$ $\angle B + \angle D = 180°$ $\Leftrightarrow$ $\angle B = \angle D = 90°$. This establishes the result.

(c) [R. Barrington Leigh] Let a, b and c be the respective lengths of the segments BC, AC and AB. Let O and I be, respectively, the circumcentre and the incentre for the quadrilateral. Note that both points lie on the diagonal AC. Wolog, we may take $a \geq c$.

We observe that $\angle ABI = 45^\circ$ and that BI is the hypotenuse of an isosceles right triangle with arms of length r. We have, by the Law of Cosines,

$$\begin{aligned} d^2 &= R^2 + 2r^2 - 2\sqrt{2}Rr \cos \angle IBO \\ &= R^2 + 2r^2 - 2\sqrt{2}Rr \left[\cos \left(\angle ABO - \frac{\pi}{4} \right) \right] \\ &= R^2 + 2r^2 - 2\sqrt{2}Rr [\cos \angle ABO (1/\sqrt{2}) + \sin \angle ABO (1/\sqrt{2})] \\ &= R^2 + 2r^2 - 2Rr [\cos \angle ABO + \cos \angle CBO] \\ &= R^2 + 2r^2 - 2Rr [\cos \angle BAC + \cos \angle BCA] \\ &= R^2 + 2r^2 - 2Rr \left(\frac{a+c}{b} \right) \\ &= R^2 + 2r^2 - r(a+c) \ , \end{aligned}$$

since $b = 2R$ and both triangle AOB and COB are isosceles with arms of length R.

By looking at the area of ΔABC in two ways, we have that $ac = r(a+c)$. Now

$$(a+c-r)^2 = a^2 + c^2 + r^2 + 2[ac - r(a+c)] = a^2 + c^2 + r^2 = 4R^2 + r^2 \ ,$$

so that $a + c = r + \sqrt{4R^2 + r^2}$. (The positive root is selected as a and c both exceed r.) Hence

$$\begin{aligned} d^2 &= R^2 + 2r^2 - r[r + \sqrt{4R^2 + r^2}] \\ &= R^2 + r^2 - r\sqrt{r^2 + 4R^2} \ . \end{aligned}$$

87. Prove that, if the real numbers a, b, c, satisfy the equation

$$\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor$$

for each positive integer n, then at least one of a and b is an integer.

Solution. We first show that $a + b = c$. Suppose, if possible, that $c > a + b$. Let $n \geq (c - a - b)^{-1}$. Then

$$\lfloor nc \rfloor > nc - 1 > n(a+b) > \lfloor na \rfloor + \lfloor nb \rfloor$$

yielding a contradiction. Similarly, if $c < a + b$ and $n \geq 2(a + b - c)^{-1}$, then

$$\lfloor nc \rfloor \leq nc \leq na - 1 + nb - 1 < \lfloor na \rfloor + \lfloor nb \rfloor$$

again yielding a contradiction. Hence $a + b = c$.

Let $a = \lfloor a \rfloor + \alpha$, $b = \lfloor b \rfloor + \beta$ and $c = \lfloor c \rfloor + \gamma$. From the condition for $n = 1$, we have $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor c \rfloor$. Then, $\lfloor na \rfloor = \lfloor n\lfloor a \rfloor + n\alpha \rfloor = n\lfloor a \rfloor + \lfloor n\alpha \rfloor$ with similar equations for b and c. Putting this together gives that $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$, for all $n \geq 1$. As in the first part of the solution, we have that $\alpha + \beta = \gamma$, from which $\{n\alpha\} + \{n\beta\} = \{n\gamma\}$ for all $n \geq 1$, where $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ of x.

We first show that α, β and γ cannot all be positive and rational. For, if they were rational, then for some positive integers i, j, k with $k \geq 2$, we would have $\alpha = i/k$, $\beta = j/k$ and $\gamma = (i+j)/k$. Then $\lfloor k\alpha \rfloor = i = \lfloor (k-1)\alpha \rfloor + 1$, with similar relations for β and γ. Thus,

$$\lfloor k\alpha \rfloor + \lfloor k\beta \rfloor - \lfloor k\gamma \rfloor - [\lfloor (k-1)\alpha \rfloor + \lfloor (k-1)\beta \rfloor - \lfloor (k-1)\gamma \rfloor] = 1$$

so that $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$ must fail for either $n = k$ or $n = k - 1$.

Wolog, suppose that all of α, β, γ are positive with α irrational. Let p be a nonnegative integer for which $\alpha + p\beta < 1 \leq \alpha + (p+1)\beta$ and suppose that $\epsilon = 1 - (\alpha + p\beta)$. Since $\alpha + \beta = \gamma < 1$, it follows that $p \geq 1$.

We show that there is a positive integer $m \geq 2$ for which $\alpha + p\beta < \{m\alpha\}$. Let $t = \lceil 1/\epsilon \rceil$ and consider the intervals $[i/t, (i+1)/t)$ where $0 \leq i \leq t-1$. By the Pigeonhole Principle, one of these intervals must contain two of the numbers $\{2\alpha\}, \{4\alpha\}, \cdots, \{2(t+1)\alpha\}$, say $\{q\alpha\}$ and $\{r\alpha\}$ with $q \geq r + 2$. Thus, $|\{q\alpha\} - \{r\alpha\}| < 1/t \leq \epsilon$. Since

$$\{q\alpha\} - \{r\alpha\} = (q-r)\alpha - \lfloor q\alpha \rfloor + \lfloor r\alpha \rfloor = (q-r)\alpha \pm I$$

for some integer I, either $\{(q-r)\alpha\} < \epsilon$ or $\{(q-r)\alpha\} > 1 - \epsilon$.

In the first case, we can find a positive integer s for which $1 > s\{(q-r)\alpha\} > 1 - \epsilon$. Since

$$s(q-r)\alpha = s\lfloor (q-r)\alpha \rfloor + s\{(q-r)\alpha\} ,$$

it follows that

$$\{s(q-r)\alpha\} = s\{(q-r)\alpha\} > 1 - \epsilon .$$

Thus, in either case, we can find $m \geq 2$ for which $\alpha + p\beta = 1 - \epsilon < \{m\alpha\}$.

Now, $\{m\alpha\} > \alpha$,

$$\{m\gamma\} = \{m\alpha\} + \{m\beta\} \geq \{m\alpha\} > \alpha + p\beta \geq \alpha + \beta = \gamma$$

and

$$\{m\beta\} = \{m\gamma\} - \{m\alpha\} < 1 - (\alpha + p\beta) = 1 - [\alpha + (p+1)\beta - \beta] \leq 1 - (1 - \beta) = \beta .$$

Hence,

$$\lfloor m\alpha \rfloor = m\alpha - \{m\alpha\} = (m-1)\alpha - (\{m\alpha\} - \alpha) < (m-1)\alpha$$

so that $\lfloor (m-1) \rfloor = \lfloor m\alpha \rfloor$;

$$\lfloor m\gamma \rfloor = m\gamma - \{m\gamma\} = (m-1)\gamma - (\{m\gamma\} - \gamma) < (m-1)\gamma$$

so that $\lfloor (m-1)\gamma \rfloor = \lfloor m\gamma \rfloor$; and $\lfloor m\beta \rfloor = m\beta - \{m\beta\} > (m-1)\beta \geq \lfloor (m-1)\beta \rfloor$, so that $\lfloor (m-1)\beta \rfloor + 1 = \lfloor m\beta \rfloor$. Hence $\lfloor n\alpha \rfloor + \lfloor n\beta \rfloor = \lfloor n\gamma \rfloor$ must fail for either $n = m$ or $n = m - 1$.

The only remaining possibility is that either α or β vanishes, *i.e.*, that a or b is an integer. This possibility is feasible when the other two variables have the same fractional part.

88. Let I be a real interval of length $1/n$. Prove that I contains no more than $\frac{1}{2}(n+1)$ irreducible fractions of the form p/q with p and q positive integers, $1 \leq q \leq n$ and the greatest common divisor of p and q equal to 1.

Comment. The statement of the problem needs a slight correction. The result does not apply for closed intervals of length $1/n$ whose endpoints are consecutive fractions with denominator n. The interval $[1/3, 2/3]$ is a counterexample. So we need to strengthen the hypothesis to exclude this case, say by requiring that the interval be open (*i.e.*, not include its endpoints), or by supposing that not both endpoints are rational. Alternatively, we could change the bound to $\frac{1}{2}(n+3)$ and ask under what circumstances this bound is achieved.

Solution 1. We first establish a lemma: *Let $1 \leq q \leq n$. Then there exists a positive integer m for which* $\frac{1}{2}(n+1) \leq mq \leq n$. For, let $m = \lfloor n/q \rfloor$. If $q > n/2$, then $m = 1$ and clearly $\frac{1}{2}(n+1) \leq mq = q \leq n$. If $q \leq n/2$, then $(n/q) - 1 < m \leq (n/q)$, so that

$$\frac{n}{2} \leq n - q < mq \leq n$$

and $\frac{1}{2}(n+1) \leq mq \leq n$.

Let p/q and p'/q' be two irreducible fractions in I with m and m' corresponding integers as determined by the lemma. Suppose, if possible, that $mq = m'q'$. Then

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = \left|\frac{mp}{mq} - \frac{m'p'}{mq}\right| \geq \frac{1}{mq} \geq \frac{1}{n} \ ,$$

contradicting the fact that no two fractions in I can be distant at least $\frac{1}{n}$.

It follows that the mapping $p/q \to mq$ from the set of irreducible fractions in I into the set of integers in the interval $[(n+1)/2, n]$ is one-one. But the latter set has at most $n - ((n+1)/2) + 1 = (n+1)/2$ elements, and the result follows.

Solution 2. [M. Zaharia] For $1 \leq i \leq \frac{1}{2}(n+1)$, define

$$S_i = \{2^j(2i-1) : j = 0, 1, 2, \cdots\} \ .$$

(Thus, $S_1 = \{1, 2, 4, 8, \cdots\}$, $S_3 = \{3, 6, 12, 24, \cdots\}$ and $S_5 = \{5, 10, 20, 40, \cdots\}$, for example.) We show that each S_i contains at most one denominator not exceeding n among the irreducible fractions in I. For suppose

$$\frac{a}{2^u(2i-1)} \quad \text{and} \quad \frac{b}{2^v(2i-1)}$$

are distinct irreducible fractions in I, with $u \leq v$. Then

$$\left|\frac{a}{2^u(2i-1)} - \frac{b}{2^v(2i-1)}\right| = \left|\frac{2^{v-u}a - b}{2^v(2i-1)}\right| \geq \frac{1}{2^v(2i-1)} \geq \frac{1}{n} \ .$$

But I cannot contain two fractions separated by a distance of $1/n$ or larger. Thus, we get a contradiction, and it follows that there cannot be more than one fraction with a denominator in each of the at most $(n+1)/2$ sets S_i. The result follows.

89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.

Solution 1. Let a, b, c be three numbers with the desired property; wolog, suppose that $2 \leq a \leq b \leq c$. Since $a|(bc+1)$, a has greatest common divisor 1 with each of b and c. Similarly, the greatest common divisor of b and c is 1. Since $ab + bc + ca + 1$ is a multiple of each of a, b, c, it follows that $ab + bc + ca + 1$ is a multiple of abc. Therefore, $abc \leq ab + bc + ca + 1$.

Since a, b, c are distinct and so $2 \leq a < b < c$, we must have $a \geq 2$ and $b \geq 3$. Suppose, if possible that $b \geq 4$, so that $c \geq 5$. Then $abc \geq 40$ and

$$ab + bc + ca + 1 \leq \frac{abc}{5} + \frac{abc}{2} + \frac{abc}{4} + 1 \leq \frac{19abc}{20} + \frac{abc}{40} < abc \ ,$$

a contradiction. Hence b must equal 3 and a must equal 2. Since $c|(ab+1)$, c must equal 7. The triple $(a, b, c) = (2, 3, 7)$ satisfies the desired condition and is the only triple that does so.

Solution 2. As in Solution 1, we show that $abc \leq ab + bc + ca + 1$, so that

$$1 \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} \ .$$

However, if $a \geq 3$, the right ride of the inequality cannot exceed $(1/3) + (1/4) + (1/5) + (1/60) = 4/5 < 1$. Hence $a = 2$. If $a = 2$ and $b \geq 4$, then $b \geq 5$ (why?) and the right side cannot exceed $(1/2) + (1/5) + (1/7) +$

$(1/70) = 6/7 < 1$. Hence $(a, b) = (2, 3)$. If now c exceeds 7, then $c \geq 11$ and the right side of the inequality cannot exceed $(1/2) + (1/3) + (1/11) + (1/66) = 31/33 < 1$. Hence $c \leq 7$, and we are now led to the solution.

Solution 3. [P. Du] As in Solution 1, we show that a, b, c are pairwise coprime and that $ab + bc + ca + 1$ is a multiple of abc. Assume $2 \leq a < b < c$. Then $abc > ac$ and $bc(a - 1) \geq bc > ac \geq a(b + 1) > ab + 1$, whence, adding these, we get $2abc - bc > ac + ab + 1$, so that $2abc > ab + bc + ca + 1$. Hence,

$$abc = ab + bc + ca + 1 = bc - (c - a)b + bc + bc - (b - a)c + 1 = 3bc - (c - a)b - (c - b)a + 1 < 3bc$$

so that $a < 3$. Hence $a = 2$. Plugging this into the equation yields

$$bc = 2b + 2c + 1 = 4c - 2(c - b) + 1 < 4c$$

so that $b < 4$. Hence $b = 3$, and we find that $6c = 6 + 5c + 1$ or $c = 7$.

Solution 4. [O. Ivrii] As before, we show that a, b and c are pairwise coprime, and take $2 \leq a < b < c$. Then $bc | ab + ac + 1$. Since $bc > ac + 1$ and $bc > ab + 1$, we have that $2bc > ab + ac + 1$. Hence $bc = ab + ac + 1$, so that $(bc + 1) + ab = 2(ab + 1) + ac$. Since a divides each of $bc + 1$, ab and ac, but is coprime with $ab + 1$, it follows that a divides 2. Hence $a = 2$ and

$$bc = 2(b + c) + 1 \Longrightarrow (b - 2)(c - 2) = 5 \Longrightarrow (b, c) = (3, 7) \ .$$

Solution 5. As above, we can take $2 \leq a < b < c$. Since

$$\frac{(ab + 1)}{c} \cdot \frac{(ca + 1)}{b} = a^2 + \frac{a}{c} + \frac{a}{b} + \frac{1}{bc} \ ,$$

we see that $(a/c) + (a/b) + (1/(bc))$ is a positive integer less than 3.

Suppose, if possible, that $(a/c) + (a/b) + (1/(bc)) = 2$. Then $ab + ac + 1 = 2bc$, whence $b(c - a) + c(b - a) = 1$, an impossibility. Hence $a(b + c) + 1 = bc$, so that

$$2 = (bc + 1) - a(b + c) \ .$$

Since both terms on the right are divisible by a, 2 must be a multiple of a. Hence $a = 2$, and we obtain $(b - 2)(c - 2) = 5$, so that $(b, c) = (3, 7)$.