Solutions and comments

61. Let $S = 1!2!3! \cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer k for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is k unique? (*Optional:* Is it possible to find such a number k that exceeds 100?)

Solution 1. Note that, for each positive integer j, $(2j-1)!(2j)! = [(2j-1)!]^2 \cdot 2j$. Hence

$$S = \prod_{j=1}^{50} [(2j-1)!]^2[2j] = 2^{50} 50! \left[\prod_{j=1}^{50} (2j-1)! \right]^2 ,$$

from which we see that $k = 50$ is the required number.

We show that $k = 50$ is the only possibility. First, k cannot exceed 100, for otherwise 101! would be a factor of $k!$ but not S, and so $S/k!$ would not even be an integer. Let $k \leq 100$. The prime 47 does not divide $k!$ for $k \leq 46$ and divides 50! to the first power. Since $S/50!$ is a square, it evidently divides S to an odd power. So $k \geq 47$ in order to get a quotient divisible by 47 to an even power. The prime 53 divides each $k!$ for $k \geq 53$ to the first power and divides $S/50!$, and so S to an even power. Hence, $k \leq 52$.

The prime 17 divides 50! and $S/50!$, and hence S to an even power, but it divides each of 51! and 52! to the third power. So we cannot have $k = 51$ or 52. Finally, look at the prime 2. Suppose that 2^{2u} is the highest power of 2 that divides $S/50!$ and that 2^v is the highest power of 2 that divides 50!; then 2^{2u+v} is the highest power of 2 that divides S. The highest power of 2 that divides 48! and 49! is 2^{v-1} and the highest power of 2 that divides 46! and 47! is 2^{v-5}. ¿From this, we deduce that 2 divides $S/k!$ to an odd power when $47 \leq k \leq 49$. The desired uniqueness of k follows.

Solution 2. Let p be a prime exceeding 50. Then p divides each of $m!$ to the first power for $p \leq m \leq 100$, so that p divides S to the even power $100 - (p-1) = 101 - p$. From this, it follows that if $53 \geq k$, p must divide $S/k!$ to an odd power.

On the other hand, the prime 47 divides each $m!$ with $47 \leq m \leq 93$ to the first power, and each $m!$ with $94 \leq m \leq 100$ to the second power, so that it divides S to the power with exponent $54 + 7 = 61$. Hence, in order that it divide $S/k!$ to an even power, we must make k one of the numbers $47, \cdots, 52$.

By an argument, similar to that used in Solution 1, it can be seen that 2 divides any product of the form $1!2! \cdots (2m-1)!$ to an even power and 100! to the power with exponent

$$\lfloor 100/2 \rfloor + \lfloor 100/4 \rfloor + \lfloor 100/8 \rfloor + \lfloor 100/16 \rfloor + \lfloor 100/32 \rfloor + \lfloor 100/64 \rfloor = 50 + 25 + 12 + 6 + 3 + 1 = 97 .$$

Hence, 2 divides S to an odd power. So we need to divide S by $k!$ which 2 divides to an odd power to get a perfect square quotient. This reduces the possibilities for k to 50 or 51. Since

$$S = 2^{99} \cdot 3^{98} \cdot 4^{97} \cdots 99^2 \cdot 100 = (2 \cdot 4 \cdots 50)(2^{49} \cdot 3^{49} \cdot 4^{48} \cdots 99)^2 = 50! \cdot 2^{50} (\cdots)^2 ,$$

$S/50!$ is a square, and so $S/51! = (S/50!) \div (51)$ is not a square. The result follows.

Solution 3. As above, $S/(50!)$ is a square. Suppose that $53 \leq k \leq 100$. Then 53 divides $k!/50!$ to the first power, and so $k!/50!$ cannot be square. Hence $S/k! = (S/50!) \div (k!/50!)$ cannot be square. If $k = 51$ or 52, then $k!/50!$ is not square, so $S/k!$ cannot be square. Suppose that $k \leq 46$. Then 47 divides $50!/k!$ to the first power, so that $50!/k!$ is not square and $S/k! = (S/50!) \times (50!/k!)$ cannot be square. If $k = 47, 48$ or 49, then $50!/k!$ is not square and so $S/k!$ is not square. Hence $S/k!$ is square if and only if $k = 50$ when $k \leq 100$.

62. Let n be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

Solution. Any prime divisor of $n! + 1$ must be larger than n, since all primes not exceeding n divide $n!$. Suppose, if possible, the result fails. Then, the only prime that can divide $n! + 1$ is $n + 1$, so that, for some positive integer r and nonnegative integer K,

$$n! + 1 = (n+1)^r = 1 + rn + Kn^2 .$$

This happens, for example, when $n = 1, 2, 4$: $1! + 1 = 2$, $2! + 1 = 3$, $4! + 1 = 5^2$. Note, however, that the desired result does hold for $n = 3$: $3! + 1 = 7$.

Henceforth, assume that n exceeds 4. If n is prime, then $n + 1$ is composite, so by our initial comment, all of its prime divisors exceed $n + 1$. If n is composite and square, then $n!$ is divisible by the four distinct integers $1, n, \sqrt{n}, 2\sqrt{n}$, while is n is composite and nonsquare with a nontrivial divisor d. then $n!$ is divisible by the four distinct integers $1, d, n/d, n$. Thus, $n!$ is divisible by n^2. Suppose, if possible, the result fails, so that $n! + 1 = 1 + rn + Kn^2$, and $1 \equiv 1 + rn \pmod{n^2}$. Thus, r must be divisible by n, and, since it is positive, must exceed n. Hence

$$(n + 1)^r \geq (n + 1)^n > (n + 1)n(n - 1) \cdots 1 > n! + 1 ,$$

a contradiction. The desired result follows.

63. Let n be a positive integer and k a nonnegative integer. Prove that

$$n! = (n + k)^n - \binom{n}{1}(n + k - 1)^n + \binom{n}{2}(n + k - 2)^n - \cdots \pm \binom{n}{n}k^n .$$

Solution 1. Recall the *Principle of Inclusion-Exclusion: Let S be a set of n objects, and let P_1, P_2, $\cdots$, P_m be m properties such that, for each object $x \in S$ and each property P_i, either x has the property P_i or x does not have the property P_i. Let $f(i, j, \cdots, k)$ denote the number of elements of S each of which has properties P_i, P_j, $\cdots$, P_k (and possibly others as well). Then the number of elements of S each having none of the properties P_1, P_2, $\cdots$, P_m is*

$$n - \sum_{1 \leq i \leq m} f(i) + \sum_{1 \leq i < j \leq m} f(i, j) - \sum_{1 \leq i < j < l \leq m} f(i, j, l) + \cdots + (-1)^m f(1, 2, \cdots, m) .$$

We apply this to the problem at hand. Note that an ordered selection of n numbers selected from among $1, 2, \cdots, n + k$ is a permutation of $\{1, 2, \cdots, n\}$ if and only if it is constrained to contain each of the numbers 1, 2, $\cdots$, n. Let S be the set of all ordered selections, and we say that a selection has property P_i iff its fails to include at least i of the numbers $1, 2, \cdots, n$ $(1 \leq i \leq n)$. The number of selections with property P_i is the product of $\binom{n}{i}$, the number of ways of choosing the i numbers not included and $(n + k - i)^n$, the number of ways of choosing entries for the n positions from the remaining $n + k - 1$ numbers. The result follows.

Solution 2. We begin with a lemma:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = \begin{cases} 0 & (0 \leq m \leq n - 1) \\ (-1)^n n! & (m = n) . \end{cases}$$

We use the convention that $0^0 = 1$. To prove this, note first that $i(i-1) \cdots (i-m) = i^{m+1} + b_m i^m + \cdots + b_1 i + b_0$ for some integers b_i. We use an induction argument on m. The result holds for each positive n and for $m = 0$, as the sum is the expansion of $(1 - 1)^n$. It also holds for $n = 1, 2$ and all relevant m. Fix $n \geq 3$. Suppose that it holds when m is replaced by k for $0 \le k \le m \le n-2$. Then

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^{m+1} &= \sum_{i=1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-m) - \sum_{k=0}^{m}b_k\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k \\
&= \sum_{i=m+1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-m) - 0 \\
&= \sum_{i=m+1}^{n}(-1)^i\frac{n!i!}{i!(n-i)!(i-m-1)!} = \sum_{j=0}^{n-m-1}(-1)^{m+1+j}\frac{n!}{(n-m-1-j)!j!} \\
&= \sum_{j=0}^{n-m-1}(-1)^{m+1}(-1)^j\frac{n(n-1)\cdots(n-m)[(n-m-1)!]}{(n-m-1-j)!j!} \\
&= (-1)^{m+1}n(n-1)\cdots(n-m)\sum_{j=0}^{n-m-1}(-1)^j\binom{n-m-1}{j} = 0\ .
\end{aligned}$$

(Note that the $j=0$ term is 1, which is consistent with the $0^0 = 1$ convention mentioned earlier.) So $\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^m = 0$ for $0 \le m \le n-1$. Now consider the case $m = n$:

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}i^n = \sum_{i=1}^{n}(-1)^i\binom{n}{i}i(i-1)\cdots(i-n+1) - \sum_{k=0}^{n-1}b_k\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^k\ .$$

Every term in the first sum vanishes except the nth and each term of the second sum vanishes. Hence $\sum_{i=1}^{n}(-1)^i\binom{n}{i}i^n = (-1)^n n!$.

Returning to the problem at hand, we see that the right side of the desired equation is equal to

$$\begin{aligned}
&(n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots + (-1)^n\binom{n}{n}(n+k-n)^n \\
&= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i+k)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n}\binom{n}{j}(n-i)^jk^{n-j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^i\binom{n}{i}\binom{n}{j}(n-i)^jk^{n-j} = \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^j \\
&= \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^j \\
&= \sum_{j=0}^{n}\binom{n}{j}k^{n-j}\sum_{i=0}^{n}(-1)^n(-1)^i\binom{n}{i}i^j\ .
\end{aligned}$$

When $0 \le j \le n-1$, the sum $\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^j = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^j$ vanishes, while, when $j = n$, it assunes the value $n!$. Thus, the right side of the given equation is equal to $\binom{n}{n}k^0n! = n!$ as desired.

Solution 3. Let $m = n+k$, so that $m \ge n$, and let the right side of the equation be denoted by R. Then

$$\begin{aligned}
R &= m^n - \binom{n}{1}(m-1)^n + \binom{n}{2}(m-2)^n - \cdots + (-1)^i\binom{n}{i}(m-i)^n + \cdots + (-1)^n\binom{n}{n}(m-n)^n \\
&= m^m\left[\sum_{j=0}^{n}(-1)^i\binom{n}{i}\right] - \binom{n}{1}m^{n-1}\left[\sum_{i=1}^{n}(-1)^i i\binom{n}{i}\right] + \binom{n}{2}m^{n-2}\left[\sum_{i=1}^{n}(-1)^i i^2\binom{n}{i}\right] + \cdots \\
&\qquad + (-1)^n\binom{n}{n}\left[\sum_{i=1}^{n}(-1)^i i^n\binom{n}{i}\right]\ .
\end{aligned}$$

Let

$$f_0(x) = (1-x)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i$$

and let

$$f_k(x) = xDf_{k-1}(x)$$

for $k \geq 1$, where Df denotes the derivative of a function f. Observe that, from the closed expression for $f_0(x)$, we can establish by induction that

$$f_k(x) = \sum_{i=0}^{n} (-1)^i i^k \binom{n}{i} x^i$$

so that $R = \sum_{k=0}^{n} (-1)^k \binom{n}{k} m^{n-k} f_k(1)$.

By induction, we establish that

$$f_k(x) = (-1)^k n(n-1)\cdots(n-k+1)x^k(1-x)^{n-k} + (1-x)^{n-k+1} g_k(x)$$

for some polynomial $g_k(x)$. This is true for $k = 1$ with $g_1(x) = 0$. Suppose if holds for $k = j$. Then

$$f'_j(x) = (-1)^j n(n-1)\cdots(n-j+1)x^{j-1}(1-x)^{n-j} - (-1)^j n(n-1)\cdots(n-j+1)(n-j)x^j(1-x)^{n-j-1} - (n-j+1)(1-x)^{n-j} g_j(x) + (1-x)^{n-j+1} g'_j(x) \ ,$$

whence

$$f_{j+1}(x) = (-1)^{j+1} n(n_1)\cdots(n-j)x^j(1-x)^{n-(j+1)} + (1-x)^{n-(j+1)+1}[(-1)^j n(n-1)\cdots(n-j+1)x^j - (n-j+1)xg_k(x) + x(1-x)g'_j(x)]$$

and we obtain the desired representation by induction. Then for $1 \leq k \leq n-1$, $f_k(1) = 0$ while $f_n(1) = (-1)^n n!$. Hence $R = (-1)^n f_n(1) = n!$.

64. Let M be a point in the interior of triangle ABC, and suppose that D, E, F are respective points on the side BC, CA, AB, which all pass through M. (In technical terms, they are *cevians.*) Suppose that the areas and the perimeters of the triangles BMD, CME, AMF are equal. Prove that triangle ABC must be equilateral.

Solution. [L. Lessard] Let the common area of the triangles BMD, CME and AMF be a and let their common perimeter be p. Let the area and perimeter of ΔAME be u and x respectively, of ΔMFB be v and y respectively, and of ΔCMD be w and z respectively.

By considering pairs of triangles with equal heights, we find that

$$\frac{AF}{FB} = \frac{a}{v} = \frac{2a+u}{v+a+w} = \frac{a+u}{a+w} \ ,$$

$$\frac{BD}{DC} = \frac{a}{w} = \frac{2a+v}{u+a+w} = \frac{a+v}{a+u} \ ,$$

$$\frac{CE}{EA} = \frac{a}{u} = \frac{2a+w}{u+a+v} = \frac{a+w}{a+v} \ .$$

¿From these three sets of equations, we deduce that

$$\frac{a^3}{uvw} = 1 \ ;$$

$$a^2 + (w-u)a - uv = 0 \ ,$$
$$a^2 + (u-w)a - vw = 0 \ ,$$
$$a^2 + (v-u)a - uw = 0 \ ;$$

whence
$$a^3 = uvw \qquad \text{and} \qquad 3a^2 = uv + vw + uw \ .$$

This means that uv, vw, uw are three positive numbers whose geometric and arithmetic means are both equal to a^2. Hence $a^2 = uv = vw = uw$, so that $u = v = w = a$. It follows that $AF = FB$, $BD = DC$, $CE = EA$, so that AD, BE and CF are medians and M is the centroid.

Wolog, suppose that $AB \geq BC \geq CA$. Since $AB \geq BC$, $\angle AEB \geq 90°$, and so $AM \geq MC$. Thus $x \geq p$. Similarly, $y \geq p$ and $p \geq z$.

Consider triangles BMD and AME. We have $BD \geq AE$, $BM \geq AM$, $ME = \frac{1}{2}BM$ and $MD = \frac{1}{2}AM$. Therefore

$$p - x = (BD + MD + BM) - (AE + ME + AM) = (BD - AE) + \frac{1}{2}(BM - AM) \geq 0$$

and so $p \geq x$. Since also $x \geq p$, we have that $p = x$. But this implies that $AM = MC$, so that $ME \perp AC$ and $AB = BC$. Since BE is now an axis of a reflection which interchanges A and C, as well as F and D, it follows that $p = z$ and $p = y$ as well. Thus, $AB = AC$ and $AC = BC$. Thus, the triangle is equilateral.

65. Suppose that XTY is a straight line and that TU and TV are two rays emanating from T for which $\angle XTU = \angle UTV = \angle VTY = 60°$. Suppose that P, Q and R are respective points on the rays TY, TU and TV for which $PQ = PR$. Prove that $\angle QPR = 60°$.

Solution 1. Let R be a rotation of $60°$ about T that takes the ray TU to TV. Then, if R transforms $Q \rightarrow Q'$ and $P \rightarrow P'$, then Q' lies on TV and the line $Q'P'$ makes an angle of $60°$ with QP. Because of the rotation, $\angle P'TP = 60°$ and $TP' = TP$, whence $TP'P$ is an equilateral triangle.

Since $\angle Q'TP = \angle TPP' = 60°$, $TV \| P'P$. Let T be the translation that takes P' to P. It takes Q' to a point Q'' on the ray TV, and $PQ'' = P'Q' = PQ$. Hence Q'' can be none other than the point R [why?], and the result follows.

Solution 2. The reflection in the line XY takes $P \rightarrow P$, $Q \rightarrow Q'$ and $R \rightarrow R'$. Triangles PQR' and $PQ'R$ are congruent and isosceles, so that $\angle TQP = \angle TQ'P = \angle TRP$ (since $PQ' = PR$). Hence $TQRP$ is a concyclic quadrilateral, whence $\angle QPR = \angle QTR = 60°$.

Solution 3. [S. Niu] Let S be a point on TU for which $SR \| XY$; observe that ΔRST is equilateral. We first show that Q lies between S and T. For, if S were between Q and T, then $\angle PSQ$ would be obtuse and $PQ > PS > PR$ (since $\angle PRS > 60° > \angle PSR$ in ΔPRS), a contradiction.

The rotation of $60°$ with centre R that takes S onto T takes ray RQ onto a ray through R that intersects TY in M. Consider triangles RSQ and RTM. Since $\angle RST = \angle RTM = 60°$, $\angle SRQ = 60° - \angle QRT = \angle TRM$ and $SR = TR$, we have that $\Delta RSQ \equiv \Delta RTM$ and $RQ = RM$. (ASA) Since $\angle QRM = 60°$, ΔRQM is equilateral and $RM = RQ$. Hence M and P are both equidistant from Q and R, and so at the intersection of TY and the right bisector of QR. Thus, $M = P$ and the result follows.

Solution 4. [H. Pan] Let Q' and R' be the respective reflections of Q and R with respect to the axis XY. Since $\angle RTR' = 120°$ and $TR = TR'$, $\angle QR'R = \angle TR'R = 30°$. Since Q, R, Q', R', lie on a circle with centre P, $\angle QPR = 2\angle QR'R = 60°$, as desired.

Solution 5. [R. Barrington Leigh] Let W be a point on TV such that $\angle WPQ = 60° = \angle WTU$. [Why does such a point W exist?] Then $WQTP$ is a concyclic quadrilateral so that $\angle QWP = 180° - \angle QTP = 60°$ and ΔPWQ is equilateral. Hence $PW = PQ = PR$.

Suppose $W \neq R$. If R is farther away from T than W, then $\angle RPT > \angle WPT > \angle WPQ = 60^\circ \Rightarrow 60^\circ > \angle TRP = \angle RWP > 60^\circ$, a contradiction. If W is farther away from T than R, then $\angle WPT > \angle WPQ = 60^\circ \Rightarrow 60^\circ > \angle RWP = \angle WRP > 60^\circ$, again a contradiction. So $R = W$ and the result follows.

Solution 6. [M. Holmes] Let the circle through T, P, Q intersect TV in N. Then $\angle QNP = 180^\circ - \angle QTP = 60^\circ$. Since $\angle PQN = \angle PTN = 60^\circ$, ΔPQN is equilateral so that $PN = PQ$. Suppose, if possible, that $R \neq N$. Then N and R are two points on TV equidistant from P. Since $\angle PNT < \angle PNQ = 60^\circ$ and ΔPNR is isosceles, we have that $\angle PNR < 90^\circ$, so N cannot lie between T and R, and $\angle PRN = \angle PNR = \angle PNT < 60^\circ$. Since $\angle PTN = 60^\circ$, we conclude that T must lie between R and N, which transgresses the condition of the problem. Hence R and N must coincide and the result follows.

Solution 7. [P. Cheng] Determine S on TU and Z on TY for which $SR \| XY$ and $\angle QRZ = 60^\circ$. Observe that $\angle TSR = \angle SRT = 60^\circ$ and $SR = RT$.

Consider triangles SRQ and TRZ. $\angle SRQ = \angle SRT - \angle QRT = \angle QRZ - \angle QRT = \angle TRZ$; $\angle QSR = 60^\circ = \angle ZTR$, so that $\Delta SRQ = \Delta TRZ$ (ASA).

Hence $RZ = RQ \Rightarrow \Delta RQZ$ is equilateral $\Rightarrow RZ = ZQ$ and $\angle RZQ = 60^\circ$. Now, both P and Z lie on the intersection of TY and the right bisector of QR, so they must coincide: $P = Z$. The result follows.

Solution 8. Let the perpendicular, produced, from Q to XY meet VT, produced, in S. Then $\angle XTS = \angle VTY = 60^\circ = \angle XTU$, from which is can be deduced that TX right bisects QS. Hence $PS = PQ = PR$, so that Q, R, S are all on the same circle with centre P.

Since $\angle QTS = 120^\circ$, we have that $\angle SQT = \angle QSR = 30^\circ$, so that QR must subtend an angle of 60° at the centre P of the circle. The desired result follows.

Solution 9. [A.Siu] Let the right bisector of QR meet the circumcircle of TQR on the same side of QR at T in S. Since $\angle QSR = \angle QTR = 60^\circ$ and $QS = QR$, $\angle SQR = \angle SRQ = 60^\circ$. Hence $\angle STQ = 180^\circ - \angle SRQ = 120^\circ$. But $\angle YTQ = 120^\circ$, so S must lie on TY. It follows that $S = P$.

Solution 10. Assign coordinates with the origin at T and the $x-$axis along XY. The the respective coordinates of Q and R have the form $(u, -\sqrt{3}u)$ and $(v, \sqrt{3}v)$ for some real u and v. Let the coordinates of P be $(w, 0)$. Then $PQ = PR$ yields that $w = 2(u + v)$. [*Exercise:* work it out.]

$$\begin{aligned} |PQ|^2 - |QR|^2 &= (u - w)^2 + 3u^2 - (u - v)^2 - 3(u + v)^2 \\ &= w^2 - 2uw - 4v(u + v) = w^2 - 2uw - 2vw \\ &= w^2 - 2(u + v)w = 0 \ . \end{aligned}$$

Hence $PQ = QR = PR$ and ΔPQR is equilateral. Therefore $\angle QPR = 60^\circ$.

Solution 11. [J.Y. Jin] Let C be the circumcircle of ΔPQR. If T lies strictly inside C, then $60^\circ = \angle QTR > \angle QPR$ and $60^\circ = \angle PTR > \angle PQR = \angle PRQ$. Thus, all three angle of ΔPQR would be less than 60°, which is not possible. Similarly, if T lies strictly outside C, then $60^\circ = \angle QTR < \angle QPR$ and $60^\circ = \angle PTR < \angle PQR = \angle PRQ$, so that all three angles of ΔPQR would exceed 60°, again not possible. Thus T must be on C, whence $\angle QPR = \angle QTR = 60^\circ$.

Solution 12. [C. Lau] By the Sine Law,

$$\frac{\sin \angle TQP}{|TP|} = \frac{\sin 120^\circ}{|PQ|} = \frac{\sin 60^\circ}{|PR|} = \frac{\sin \angle TRP}{|TP|} \ ,$$

whence $\sin \angle TQP = \sin \angle TRP$. Since $\angle QTP$, in triangle QTP is obtuse, $\angle TQP$ is acute.

Suppose, if possible, that $\angle TRP$ is obtuse. Then, in triangle TPR, TP would be the longest side, so $PR < TP$. But in triangle TQP, PQ is the longest side, so $PQ > TP$, and so $PQ \neq PR$, contrary to hypothesis. Hence $\angle TRP$ is acute. Therefore, $\angle TQP = \angle TRP$. Let PQ and RT intersect in Z. Then, $60^\circ = \angle QTZ = 180^\circ - \angle TQP - \angle QZT = 180^\circ - \angle TRP - \angle RZP = \angle QPR$, as desired.

66. (a) Let $ABCD$ be a square and let E be an arbitrary point on the side CD. Suppose that P is a point on the diagonal AC for which $EP \perp AC$ and that Q is a point on AE produced for which $CQ \perp AE$. Prove that B, P, Q are collinear.

(b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?

Solution 1. Let $ABCD$ be a rectangle, and let E, P, Q be determined as in the problem. Suppose that $\angle ACD = \angle BDC = \alpha$. Then $\angle PEC = 90^\circ - \alpha$. Because $EPQC$ is concyclic, $\angle PQC = \angle PEC = 90^\circ - \alpha$. Because $ABCQD$ is concyclic, $\angle BQC = \angle BDC = \alpha$. The points B, P, Q are collinear $\Longleftrightarrow \angle BQC = \angle PQC \Leftrightarrow \alpha = 90^\circ - \alpha \Leftrightarrow \alpha = 45^\circ \Leftrightarrow ABCD$ is a square.

Solution 2. (a) $EPQC$, with a pair of supplementary opposite angles, is concyclic, so that $\angle CQP = \angle CEP = 180^\circ - \angle EPC - \angle ECP = 45^\circ$. Since $CBAQ$ is concyclic, $\angle CQB = \angle CAB = 45^\circ$. Thus, $\angle CQP = \angle CQB$ so that Q, P, B are collinear.

(b) Suppose that $ABCD$ is a nonquare rectangle. Then taking $E = D$ yields a counterexample.

Solution 3. (a) The circle with diameter AC that passes through the vertices of the square also passes through Q. Hence $\angle QBC = \angle QAC$. Consider triangles PBC and EAC. Since triangles ABC and EPC are both isosceles right triangles, $BC : AC = PC : EC$. Also $\angle BCA = \angle PCE = 45^\circ$. Hence $\Delta PBC \sim \Delta EAC$ (SAS) so that $\angle PBC = \angle EAC = \angle QAC = \angle QBC$. It follows that Q, P, B are collinear.

Solution 4. [S. Niu] Let $ABCD$ be a rectangle and let E, P, Q be determined as in the problem. Let EP be produced to meet BC in F. Since $\angle ABF = \angle APF$, the quadrilateral $ABPF$ is concyclic, so that $\angle PBC = \angle PBF = \angle PAF$. Since $ABCQ$ is concyclic, $\angle QBC = \angle QAC = \angle PAE$. Now B, P, Q are collinear

$$\Leftrightarrow \angle PBC = \angle QBC \Leftrightarrow \angle PAF = \angle PAE \Leftrightarrow AC \ \text{ right bisects } \ EF$$

$$\Leftrightarrow \angle ECA = \angle ACB = 45^\circ \Leftrightarrow ABCD \ \text{ is a square} .$$

Solution 5. [M. Holmes] (a) Suppose that BQ intersects AC in R. Since $ABCQD$ is concyclic, $\angle AQR = \angle AQB = \angle ACB = 45^\circ$, so that $\angle BQC = 45^\circ$. Since $\angle EQR = \angle AQB = \angle ECR = 45^\circ$, $ERCQ$ is concyclic, so that $\angle ERC = 180^\circ - \angle EQC = 90^\circ$. Hence $ER \perp AC$, so that $R = P$ and the result follows.

Solution 6. [L. Hong] (a) Let QC intersect AB in F. We apply Menelaus' Theorem to triangle AFC: B, P, Q are collinear if and only if

$$\frac{AB}{BF} \cdot \frac{FQ}{QC} \cdot \frac{CP}{PA} = -1 .$$

Let the side length of the square be 1 and the length of DE be a. Then $|AB| = 1$. Since $\Delta ADE \sim \Delta FBC$, $AD : DE = BF : BC$, so that $|BF| = 1/a$ and $|FC| = \sqrt{1+a^2}/a$. Since $\Delta ADE \sim \Delta CQE$, $CQ : EC = AD : EA$, so that $|CQ| = (1-a)/\sqrt{1+a^2}$. Hence

$$\frac{|FQ|}{|CQ|} = 1 + \frac{|FC|}{|CQ|} = 1 + \frac{1+a^2}{a(1-a)} = \frac{1+a}{a(1-a)} .$$

Since ΔECP is right isosceles, $|CP| = (1-a)/\sqrt{2}$ and $|PA| = \sqrt{2} - |CP| = (1+a)/\sqrt{2}$. Hence $|CP|/|PA| = (1-a)/(1+a)$. Multiplying the three ratios together and taking account of the directed segments gives the product -1 and yields the result.

Solution 7. (a) Select coordinates so that $A \sim (0,1)$, $B \sim (0,0)$, $C \sim (1,0)$, $D \sim (1,1)$ and $E \sim (1,t)$ for some t with $0 \le t \le 1$. It is straightforward to verify that $P \sim (1 - \frac{t}{2}, \frac{t}{2})$.

Since the slope of AE is $t-1$, the slope of AQ should be $(1-t)^{-1}$. Since the coordinates of Q have the form $(1+s, s(1-t)^{-1})$ for some s, it is straightforward to verify that

$$Q \sim \left(\frac{2-t}{1+(1-t)^2}, \frac{t}{1+(1-t)^2)} \right) .$$

It can now be checked that the slope of each of BQ and BP is $t(2-t)^{-1}$, which yields the result.

(b) The result fails if $A \sim (0,2)$, $B \sim (0,0)$, $C \sim (1,0)$, $D \sim (1,2)$. If $E \sim (1,1)$, then $P \sim (\frac{3}{5}, \frac{4}{5})$ and $Q \sim (\frac{3}{2}, \frac{1}{2})$.